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AN

ELEMENTARY TREATISE

ON

CONIC SECTIONS

BY

CHARLES SMITH, M.A.,

MASTER OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE.

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PREFACE TO THE FIRST EDITION.

In the following work I have investigated the more elementary properties of the Ellipse, Parabola, and Hyperbola, defined with reference to a focus and directrix, before considering the General Equation of the Second Degree. I believe that this arrangement is the best for beginners.

The examples in the body of each chapter are for the most part very easy applications of the book-work, and have been carefully selected and arranged to illustrate the principles of the subject. The examples at the end of each chapter are more difficult, and include very many of those which have been set in the recent University and College examinations, and in the examinations for Open Scholarships, in Cambridge.

The answers to the examples, together with occasional hints and solutions, are given in an appendix. I have also, in the body of the work, given complete solutions of some illustrative examples, which I hope will be found especially useful.

Although I have endeavoured to present the elementary parts of the subject in as simple a manner as possible for the benefit of beginners, I have tried to make the work in some degree complete; and have therefore included a chapter on Trilinear Co-ordinates, and short accounts of the methods of Reciprocation and Conical Projection. For fuller information on these latter subjects the student should consult the works of Dr Salmon, Dr Ferrers, and Dr C. Taylor, to all of whom it will be seen that I am largely indebted.

I am indebted to several of my friends for their kindness in looking over the proof sheets, for help in the verification of the examples, and for valuable suggestions; and it is hoped that few mistakes have escaped detection.

CHARLES SMITH.

SIDNEY SUSSEX COLLEGE,
April, 1882.

PREFACE TO THE SECOND EDITION.

THE second edition has been carefully revised, and some additions have been made, particularly in the last Chapter.

SIDNEY SUSSEX COLLEGE,
July, 1883.

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[*The Articles marked with an asterisk may be omitted by beginners until after they have read Chapter IX.*]

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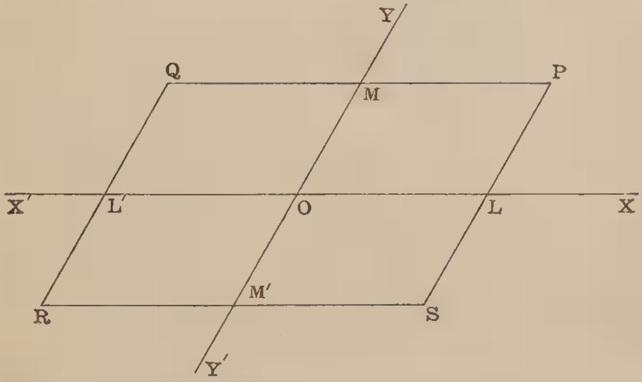
CHAPTER I.

CO-ORDINATES.

1. IF in a plane two fixed straight lines XOX', YOY' be taken, and through any point P in the plane the two straight lines PM, PL be drawn parallel to XOX', YOY' respectively; the position of the point P can be found when the lengths of the lines PM, PL are given. For we have only to take OL, OM equal respectively to the known lines PM, PL and complete the parallelogram $LOMP$.



The lengths MP and LP, or OL and OM, which thus define the position of the point P with reference to the lines OX, OY are called the *co-ordinates* of the point P with reference to the *axes* OX, OY. The point of intersection of the *axes* is called the *origin*. When the angle between the axes is a right angle the axes are said to be *rectangular*; when the angle between the axes is not a right angle the axes are said to be *oblique*.

OL is generally called the *abscissa*, and LP the *ordinate* of the point P.

The co-ordinate which is measured along the axis OX is denoted by the letter x, and that measured along the axis OY by the letter y. If, in the figure, OL be a and OM be b; then at the point P, $x=a$, and $y=b$, and the point is for shortness often called the point (a, b).

2. Let OM' be taken equal to OM, and OL' equal to OL, and through M', L' draw lines parallel to the axes, as in the figure to Art. 1. Then the co-ordinates of the three points Q, R, S will be equal in magnitude to those of P. Hence it is not sufficient to know the lengths of the lines OL, LP, we must also know the directions in which they are measured.

If lines measured in one direction be taken as positive, lines measured in the opposite direction must be taken as negative. We shall consider lines measured in the directions OX or OY to be positive, those therefore in the directions OX' or OY' must be considered negative.

We are now able to distinguish between the co-ordinates of the points P, Q, R, S. The co-ordinates of R are OL', $L'R$, and these are both measured in the negative direction; so that, if the co-ordinates of P be a, b, those of R will be $-a$, $-b$. The co-ordinates of S will be a, $-b$; and those of Q will be $-a$, b.

3. It must be carefully noticed that whether a line is positive or negative depends on the *direction* in which it is measured, and does not depend on the position of the origin; for example, in the figure to Art. 1, the line LO is negative although the line OL is positive.

If any two points K, L be taken and the distances

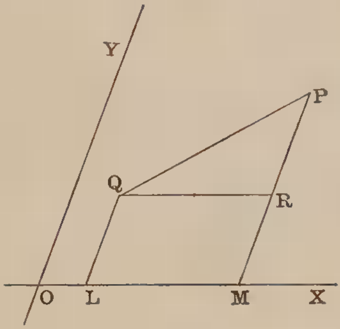
OK, OL, measured from a point O in the line KL, be a and b respectively, then the distance KL must be $KO+OL$, or $-OK+OL$, that is $-a+b$, and this will be the case wherever the point O, from which distances are measured, may be.

If $OA=-3$, and $OB=4$; then $AB=-(-3)+4=7$. If $OA=3$, and $OB=-4$; then $AB=-3+(-4)=-7$.

The reader may illustrate this by means of a figure.

4. *To express the distance between two points in terms of their co-ordinates.*

Let P be the point (x', y'), and Q the point (x'', y''), and let the axes be inclined at an angle ω.



Draw PM, QL parallel to OY, and QR parallel to OX, as in the figure.

Then $OL=x''$, $LQ=y''$, $OM=x'$, $MP=y'$.

By trigonometry

$$PQ^2=QR^2+RP^2-2\,QR\,.\,RP\cos QRP.$$

But

$$QR=LM=OM-OL=x'-x'',$$

$$RP=MP-MR=MP-LQ=y'-y'',$$

and angle $QRP=$ angle $OMP=\pi-$ angle $XOY=\pi-\omega$,

$$\therefore\quad PQ^2=(x'-x'')^2+(y'-y'')^2+2(x'-x'')(y'-y'')\cos\omega,$$

or $PQ=\pm\sqrt{\{(x'-x'')^2+(y'-y'')^2+2(x'-x'')(y'-y'')\cos\omega\}}$.

If the axes be at right angles to one another we have

$$PQ=\pm\sqrt{\{(x'-x'')^2+(y'-y'')^2\}}.$$

The distance of P from the origin can be obtained from the above by putting $x''=0$ and $y''=0$. The result is

$$OP = \pm\sqrt{\{(x'^2 + y'^2 + 2x'y'\cos\omega\}},$$

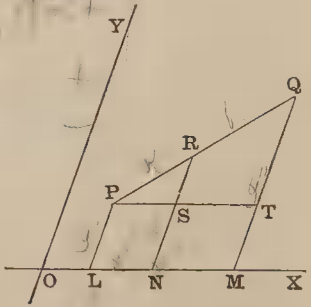
or, if the axes be rectangular,

$$OP = \pm\sqrt{\{x'^2 + y'^2\}}.$$

Except in the case of straight lines parallel to one of the axes, no convention is made with regard to the direction which is to be considered positive. We may therefore suppose either PQ or QP to be positive. If however we have three or more points P, Q, R... in the same straight line, we must consider the same direction as positive throughout, so that in all cases we must have $PQ + QR = PR$.

5. *To find the co-ordinates of a point which divides in a given ratio the straight line joining two given points.*

Let the co-ordinates of P be x_1, y_1, and the co-ordinates of Q be x_2, y_2, and let R (x, y) be the point which divides PQ in the ratio $k : l$.



Draw PL, RN, QM parallel to the axis of y, and PST parallel to the axis of x, as in the figure.

Then $\quad LN : NM :: PS : ST :: PR : RQ :: k : l;$

$$\therefore \quad l\,.\,LN - k\,.\,NM = 0,$$

or $\quad l\,(x - x_1) - k\,(x_2 - x) = 0;$

$$\therefore \quad x = \frac{lx_1 + kx_2}{l+k}.$$

Similarly $$y = \frac{ly_1 + ky_2}{l+k}.$$

The most useful case is when the line PQ is bisected: the co-ordinates of the point of bisection are

$$\tfrac{1}{2}(x_1 + x_2),\ \tfrac{1}{2}(y_1 + y_2).$$

If the line were cut *externally* in the ratio $k : l$ we should have

$$LN : MN :: k : l,$$

or $$LN : NM :: k : -l,$$

and therefore $x = \dfrac{kx_2 - lx_1}{k-l}$, $y = \dfrac{ky_2 - ly_1}{k-l}$.

The above results are true whatever the angle between the co-ordinate axes may be. But in most cases formulae become more complicated when the axes are not at right angles to one another. *We shall in future consider the axes to be at right angles in all cases except when the contrary is expressly stated.*

Ex. 1. Mark in a figure the position of the point $x=1$, $y=2$, and of the point $x=-3$, $y=-1$; and shew that the distance between them is 5.

Ex. 2. Find the lengths of the lines joining the following pairs of points: (i) $(1, -1)$ and $(-1, 1)$; (ii) $(a, -a)$ and $(-b, b)$; (iii) $(3, 4)$ and $(-1, 1)$.

Ex. 3. Shew that the three points $(1, 1)$, $(-1, -1)$ and $(-\sqrt{3}, \sqrt{3})$, are the angular points of an equilateral triangle.

Ex. 4. Shew that the four points $(0, -1)$, $(-2, 3)$, $(6, 7)$ and $(8, 3)$ are the angular points of a rectangle.

Ex. 5. Mark in a figure the positions of the points $(0, -1)$, $(2, 1)$, $(0, 3)$ and $(-2, 1)$, and shew that they are at the corners of a square.

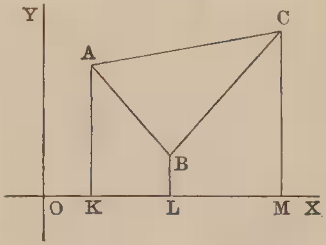
Shew the same of the points $(2, 1)$, $(4, 3)$, $(2, 5)$ and $(0, 3)$.

Ex. 6. Shew that the four points $(2, 1)$, $(5, 4)$, $(4, 7)$ and $(1, 4)$ are the angular points of a parallelogram.

Ex. 7. If the point (x, y) be equidistant from the two points $(3, 4)$ and $(1, -2)$, then will $x + 3y = 5$.

6. *To express the area of a triangle in terms of the co-ordinates of its angular points.*

Let the co-ordinates of the angular points A, B, C be x_1, y_1; x_2, y_2; and x_3, y_3 respectively.



Draw the lines AK, BL, CM parallel to the axis of y, as in the figure.

$$\triangle ABC = KACM - MCBL - LBAK.$$

Now
$$\begin{aligned} KACM &= \triangle ACM + \triangle AKM \\ &= \tfrac{1}{2} KM \,.\, MC + \tfrac{1}{2} KM \,.\, KA \\ &= \tfrac{1}{2}(x_3 - x_1)(y_3 + y_1). \end{aligned}$$

Similarly $MCBL \qquad = \frac{1}{2}(x_3 - x_2)(y_3 + y_2)$,

and $LBAK \qquad = \frac{1}{2}(x_2 - x_1)(y_2 + y_1)$;

$$\therefore \triangle ABC = \tfrac{1}{2}\{(y_3 + y_1)(x_3 - x_1) + (y_3 + y_2)(x_2 - x_3) + (y_2 + y_1)(x_1 - x_2)\};$$

or, omitting the terms which cancel,

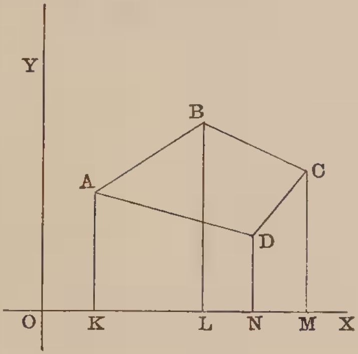
$$\triangle ABC = \tfrac{1}{2}\{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\}$$

$$= \frac{1}{2}\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}.$$

The above expression for the area of a triangle will be found to be positive if the order of the angular points be such that in going round the triangle the area is always on the left hand. Whenever on substitution a negative result for the area is obtained, a reverse order of proceeding round the triangle has been adopted.

7. *To express the area of a quadrilateral in terms of the co-ordinates of its angular points.*

Let the angular points A, B, C, D, taken in order, be (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4).



Draw AK, BL, CM, DN parallel to the axis of y, as in the figure.

Then the area $ABCD$

$$= KABL + LBCM - MCDN - NDAK.$$

And, as in the preceding Article,

$$\begin{aligned} KABL &= \tfrac{1}{2}(y_1+y_2)(x_2-x_1), \\ LBCM &= \tfrac{1}{2}(y_2+y_3)(x_3-x_2), \\ MCDN &= \tfrac{1}{2}(y_3+y_4)(x_3-x_4), \\ NDAK &= \tfrac{1}{2}(y_4+y_1)(x_4-x_1). \end{aligned}$$

Hence $ABCD = \frac{1}{2}\{(y_1+y_2)(x_2-x_1)+(y_2+y_3)(x_3-x_2)$
$\qquad + (y_3+y_4)(x_4-x_3)+(y_4+y_1)(x_1-x_4)\}$;

or, omitting the terms which cancel,

$$ABCD = \tfrac{1}{2}\{y_1x_2 - y_2x_1 + y_2x_3 - y_3x_2 + y_3x_4 - y_4x_3 + y_4x_1 - y_1x_4\}.$$

The area of any polygon may be found in a similar manner.

Ex. 1. Find the area of the triangle whose angular points are (2, 1), (4, 3) and (2, 5).

Also find the area of the triangle whose angular points are (4, −5), (5, −6) and (3, 1). Ans. 4, $\frac{5}{2}$.

Ex. 2. Find the area of the quadrilateral whose angular points are (1, 2), (3, 4), (5, 3) and (6, 2).

Also of the quadrilateral whose angular points are (2, 2), (−2, 3), (−3, −3) and (1, −2). Ans. $\frac{11}{2}$, 20.

8. If a curve be defined geometrically by a property common to all points of it, there will be some algebraical relation which is satisfied by the co-ordinates of all points of the curve, and by the co-ordinates of no other points. This algebraical relation is called the *equation of the curve.*

Conversely all points whose co-ordinates satisfy a given algebraical equation lie on a curve which is called the *locus* of that equation.

For example, if a straight line be drawn parallel to the axis OY and at a distance a from it, the abscissae of points on this line are all equal to the constant quantity a, and the abscissa of no other point is equal to a.

Hence $x=a$ is the equation of the line.

Conversely the line drawn parallel to the axis of y and at a distance a from it is the *locus* of the equation $x=a$.

Again, if x, y be the co-ordinates of any point P on a circle whose centre is the origin O and whose radius is equal to c, the square of the distance OP will be equal to x^2+y^2 [Art. 4]. But OP is equal to the radius of the circle. Therefore the co-ordinates x, y of any point on the circle satisfy the relation $x^2+y^2=c^2$. That is, $x^2+y^2=c^2$ is the *equation* of the circle.

Conversely the *locus* of the equation $x^2+y^2=c^2$ is a circle whose centre is the origin and whose radius is equal to c.

In Analytical Geometry we have to find the equation which is satisfied by the co-ordinates of all the points on a curve which has been defined by some geometrical property; and we have also to find the position and deduce the geometrical properties of a curve from the equation which is satisfied by the co-ordinates of all the points on it.

An equation is said to be of the n^{th} degree when, after it has been so reduced that the indices of the variables are the smallest possible integers, the term or terms of highest dimensions is of n dimensions. For example, the equations $axy + bx + c = 0$, $x^2 + xy\sqrt{a} + b^3 = 0$, and $\sqrt{x} + \sqrt{y} = 1$ are all of the second degree.

Ex. 1. A point moves so that its distances from the two points, (3, 4), and (5, – 2) are equal to one another; find the equation of its locus.

Ans. $x - 3y = 1$.

Ex. 2. A point moves so that the sum of the squares of its distances from the two fixed points $(a, 0)$ and $(-a, 0)$ is constant $(2c^2)$; find the equation of its locus. Ans. $x^2 + y^2 = c^2 - a^2$.

Ex. 3. A point moves so that the difference of the squares of its distances from the two fixed points $(a, 0)$ and $(-a, 0)$ is constant (c^2); find the equation of its locus. Ans. $4ax = \pm c^2$.

Ex. 4. A point moves so that the ratio of its distances from two fixed points is constant; find the equation of its locus.

Ex. 5. A point moves so that its distance from the axis of x is half its distance from the origin; find the equation of its locus.

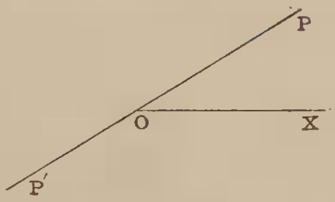
Ans. $3y^2 - x^2 = 0$.

Ex. 6. A point moves so that its distance from the axis of x is equal to its distance from the point (1, 1); find the equation of its locus.

Ans. $x^2 - 2x - 2y + 2 = 0$.

9. The position of a point on a plane can be defined by other methods besides the one described in Art. 1. A useful method is the following.

If an origin O be taken, and a fixed line OX be drawn through it; the position of any point P will be known, if the angle XOP and the distance OP be given.



These are called the *polar co-ordinates* of the point P. The length OP is called the *radius vector*, and is usually denoted by r, and the angle XOP is called the vectorial angle, and is denoted by θ.

The angle is considered to be positive if measured from OX contrary to the direction in which the hands of a watch revolve.

The radius vector is considered positive if measured from O along the line bounding the vectorial angle, and negative if measured in the opposite direction.

If PO be produced to P', so that OP' is equal to OP in magnitude, and if the co-ordinates of P be r, θ, those of P' will be either r, $\pi + \theta$ or $-r$, θ.

10. *To find the distance between two points whose polar co-ordinates are given.*

Let the co-ordinates of the two points P, Q be r_1, θ_1; and r_2, θ_2.

Then, by Trigonometry,

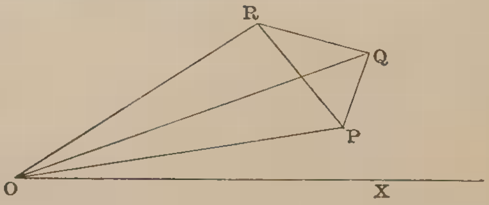
$$PQ^2 = OP^2 + OQ^2 - 2OP \, . \, OQ \cos POQ.$$

But $OP = r_1$, $OQ = r_2$ and $\angle POQ = \angle XOQ - \angle XOP = \theta_2 - \theta_1$;

$$\therefore \qquad PQ^2 = r_1^{\,2} + r_2^{\,2} - 2r_1 r_2 \cos(\theta_2 - \theta_1).$$

The polar equation of a circle whose centre is at the point (a, α) and whose radius is c, is $c^2 = a^2 + r^2 - 2ar\cos(\theta - \alpha)$; where r, θ are the polar co-ordinates of any point on it.

11. *To find the area of a triangle having given the polar co-ordinates of its angular points.*



Let P be (r_1, θ_1), Q be (r_2, θ_2), and R be (r_3, θ_3).

Then area of triangle $PQR = \triangle POQ + \triangle QOR - \triangle POR$,

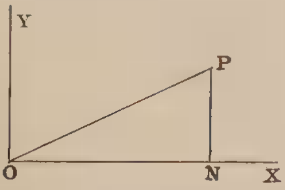
and $$\triangle POQ = \tfrac{1}{2} OP \,.\, OQ \sin POQ$$
$$= \tfrac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1),$$

so $$\triangle QOR = \tfrac{1}{2} r_2 r_3 \sin(\theta_3 - \theta_2),$$

and $$\triangle POR = \tfrac{1}{2} r_3 r_1 \sin(\theta_3 - \theta_1)$$
$$= -\tfrac{1}{2} r_3 r_1 \sin(\theta_1 - \theta_3);$$

$$\therefore \triangle PQR = \tfrac{1}{2}\{r_1 r_2 \sin(\theta_2 - \theta_1) + r_2 r_3 \sin(\theta_3 - \theta_2) + r_3 r_1 \sin(\theta_1 - \theta_3)\}.$$

12. *To change from rectangular to polar co-ordinates.*



If through O a line be drawn perpendicular to OX, and OX, OY be taken for axes of rectangular co-ordinates, we have at once

$$x = ON = OP \cos XOP = r \cos\theta,$$

and $$y = NP = OP \sin XOP = r \sin\theta.$$

Ex. 1. What are the rectangular co-ordinates of the points whose polar co-ordinates are $\left(1, \frac{\pi}{2}\right)$, $\left(2, \frac{\pi}{3}\right)$ and $\left(-4, -\frac{\pi}{4}\right)$ respectively?

Ex. 2. What are the polar co-ordinates of the points whose rectangular co-ordinates are $(-1, -1)$, $(-1, \sqrt{3})$ and $(3, -4)$ respectively?

Ex. 3. Find the distance between the points whose polar co-ordinates are $(2, 40^\circ)$ and $(4, 100^\circ)$ respectively.

Ex. 4. Find the area of the triangle the polar co-ordinates of whose angular points are $(1, 0)$, $\left(1, \frac{\pi}{2}\right)$ and $\left(\sqrt{2}, \frac{\pi}{4}\right)$ respectively.

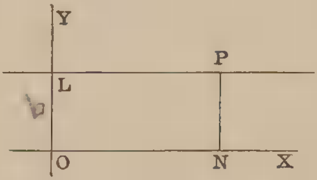
13 to 30
inclusive

CHAPTER II.

THE STRAIGHT LINE.

13. *To find the equation of a straight line parallel to one of the co-ordinate axes.*

Let LP be a straight line parallel to the axis of x and meeting the axis of y at L, and let $OL = b$.



Let x, y be the co-ordinates of any point P on the line.

Then the ordinate NP is equal to OL.

y Hence $y = b$ is the equation of the line.

Similarly $x = a$ is the equation of a straight line parallel to the axis of y and at a distance a from it.

14. *To find the equation of a straight line which passes through the origin.*

Let OP be a straight line through the origin, and let the tangent of the angle $XOP = m$.

Let x, y be the co-ordinates of any point P on the line.

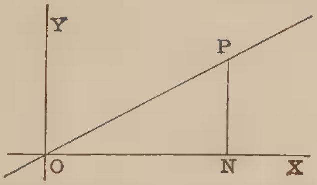
x b is y from

$NP : \frac{NP}{[illegible]}$

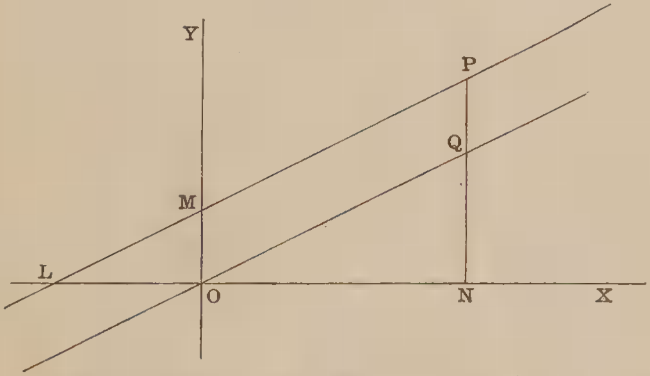
Then $\qquad NP = \tan NOP \,.\, ON.$

Hence $\qquad y = mx$ is the required equation.

Tan [illegible] p = m



15. *To find the equation of any straight line.*



Let LMP be the straight line meeting the axes in the points L, M.

Let $OM = c$, and let $\tan OLM = m$.

Let x, y be the co-ordinates of any point P on the line.

Draw PN parallel to the axis of y, and OQ parallel to the line LMP, as in the figure.

Then
$$NP = NQ + QP$$
$$= ON \tan NOQ + OM.$$

But

$NP = y$, $ON = x$, $OM = c$, and $\tan NOQ = \tan OLM = m$.

$$\therefore \qquad y = mx + c \text{.......................(i)}$$

which is the required equation.

So long as we consider any particular straight line the quantities m and c remain the same, and are therefore called constants. Of these, m is the tangent of the angle between the positive direction of the axis of x and the part of the line above the axis of x, and c is the intercept on the axis of y.

By giving suitable values to the constants m and c the equation $y=mx+c$ may be made to represent any straight line whatever. For example, the straight line which cuts the axis of y at unit distance from the origin, and makes an angle of 45° with the axis of x, has for equation $y=x+1$.

We see from (i) that the equation of any straight line is of the first degree.

16. *To shew that every equation of the first degree represents a straight line.*

The most general form of the equation of the first degree is

$$Ax+By+C=0 \quad \text{.....................(i)}$$

To prove that this equation represents a straight line, it is sufficient to shew that, if *any three* points on the locus be joined, the area of the triangle so formed will be zero.

Let (x', y'), (x'', y''), and (x''', y''') be any three points on the locus, then the co-ordinates of these points will satisfy the equation (i).

We therefore have

$$\begin{aligned} Ax' + By' + C &= 0, \\ Ax'' + By'' + C &= 0, \\ Ax''' + By''' + C &= 0. \end{aligned}$$

Eliminating A, B, C we obtain

$$\begin{vmatrix} x', & y', & 1 \\ x'', & y'', & 1 \\ x''', & y''', & 1 \end{vmatrix} = 0,$$

the area of the triangle is therefore zero [Art. 6].

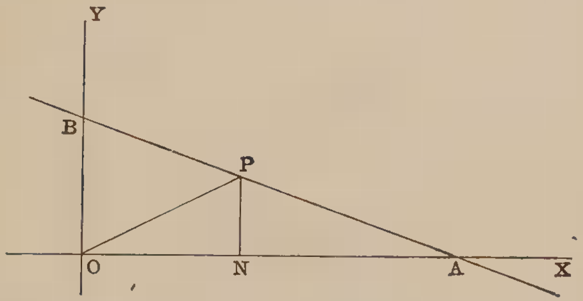
The equation $Ax+By+C=0$ is therefore the equation of a straight line.

17. The equation $Ax + By + C = 0$ appears to involve *three* constants, whereas the equation found in Art. 15 only involves *two*. But if the co-ordinates x, y of any point satisfy the equation $Ax + By + C = 0$, they will also satisfy the equation when we multiply or divide throughout by any constant. If we divide by B, we can write the equation $y = -\frac{A}{B}x - \frac{C}{B}$, and we have only the *two* constants $-\frac{A}{B}$ and $-\frac{C}{B}$ which correspond to m and c in the equation $y = mx + c$.

18. *To find the equation of a straight line in terms of the intercepts which it makes on the axes.*

Let A, B be the points where the straight line cuts the axes, and let $OA = a$, and $OB = b$.

Let the co-ordinates of any point P on the line be x, y.



Draw PN parallel to the axis of y, and join OP.

Then $$\triangle APO + \triangle PBO = \triangle ABO;$$

$$\therefore \quad ay + bx = ab,$$

or $$\frac{x}{a} + \frac{y}{b} = 1.$$

This equation may be written in the form

$$lx + my = 1,$$

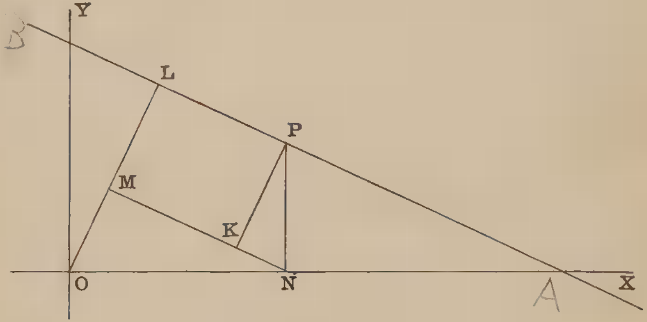
where l and m are the reciprocals of the intercepts on the axes.

19. *To find the equation of a straight line in terms of the length of the perpendicular upon it from the origin and the angle which that perpendicular makes with an axis.*

Let OL be the perpendicular upon the straight line AB, and let $OL = p$, and let the angle $XOL = \alpha$.

Let the co-ordinates of any point P on the line be x, y.

Draw PN parallel to the axis of y, NM perpendicular to OL, and PK perpendicular to NM, as in the figure.



Then,
$$OL = OM + ML = OM + KP$$
$$= ON \cos \alpha + NP \sin \alpha ;$$
or
$$p = x \cos \alpha + y \sin \alpha,$$
which is the required equation.

20. In Articles 15, 18 and 19 we have found, by independent methods, the equation of a straight line involving different constants. Any one form of the equation may however be deduced from any other.

For example, if we know the equation in terms of the intercepts on the axes, we can find the equation in terms of p and α from the relations $a \cos \alpha = p$ and $b \sin \alpha = p$, which we obtain at once from the figure to Art. 19. Hence substituting these values of a and b in the equation $\frac{x}{a}+\frac{y}{b}=1$, we get $x\cos\alpha+y\sin\alpha=p$.

If the equation of a straight line be

$$Ax+By+C=0\,;$$

then, by dividing throughout by $\sqrt{A^2+B^2}$, we have

$$\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y+\frac{C}{\sqrt{A^2+B^2}}=0.$$

Now $\frac{A}{\sqrt{A^2+B^2}}$ and $\frac{B}{\sqrt{A^2+B^2}}$ are the cosine and sine respectively of some angle, since the sum of their squares is equal to unity. If we call this angle α, we have

$$x\cos\alpha+y\sin\alpha-p=0,$$

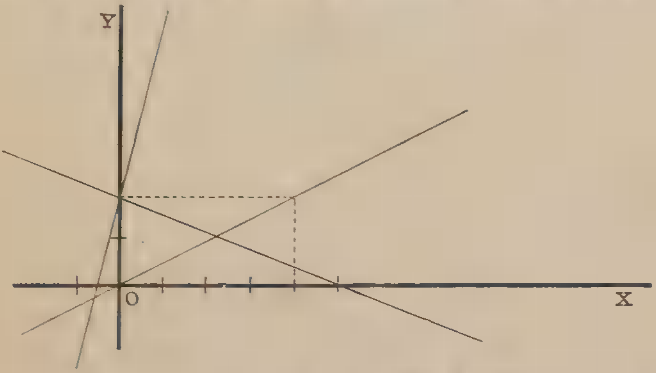
where p is put for $-\frac{C}{\sqrt{A^2+B^2}}$.

Ex. 1. If $3x-4y-5=0$, then dividing by $\sqrt{3^2+4^2}$ we have $\frac{3}{5}x-\frac{4}{5}y-1=0$. This is of the form $x\cos\alpha+y\sin\alpha-p=0$, where $\cos\alpha=\frac{3}{5}$, $\sin\alpha=-\frac{4}{5}$, and $p=1$.

Ex. 2. The equation $x+y+5=0$, is equivalent to

$$x\cos\frac{5\pi}{4}+y\sin\frac{5\pi}{4}=\frac{5}{\sqrt{2}}.$$

21. To find the position of a straight line whose equation is given, it is only necessary to find the co-ordinates of any two points on it. To do this we may give



to x any two values whatever, and find from the given equation the two corresponding values of y. The points where the line cuts the axes are easiest to find.

Ex. 1. If the equation of a line be $2x+5y=10$. Where this cuts the axis of x, $y=0$, and then $x=5$. Where it cuts the axis of y, $x=0$, and $y=2$.

Ex. 2. The intercepts made on the axes by the line $4x-y+2=0$ are $-\frac{1}{2}$, and 2 respectively.

Ex. 3. $x-2y=0$. Here the origin $(0, 0)$ is on the line, and when $x=4$, $y=2$.

The lines are marked in the figure.

22. If we wish to find the equation of a straight line which satisfies any two conditions, we may take for its equation any one of the general forms.

$$\text{(i)}\quad y=mx+c, \qquad \text{(ii)}\quad \frac{x}{a}+\frac{y}{b}=1,$$

$$\text{(iii)}\quad lx+my=1, \qquad \text{(iv)}\quad x\cos\alpha+y\sin\alpha-p=0,$$

or $\text{(v)}\quad Ax+By+C=0.$

We have then to determine the values of the two constants m and c, or a and b, or l and m, or α and p, or $\frac{A}{C}$ and $\frac{B}{C}$ for the line in question from the *two* conditions which the line has to satisfy.

Ex. 1. Find the equation of a straight line which passes through the point $(2, 3)$ and makes equal intercepts on the axes.

Let $\frac{x}{a}+\frac{y}{b}=1$ be the equation of the line.

Then, since the intercepts are equal to one another, $a=b$.

Also, since the point $(2, 3)$ is on the line,

$$\frac{2}{a}+\frac{3}{a}=1;$$

$\therefore a=5=b$ and the equation required is $\frac{x}{5}+\frac{y}{5}=1$.

Ex. 2. Find the equation of the straight line which passes through the point $(\sqrt{3}, 2)$ and which makes an angle of 60° with the axis of x.

Let $y=mx+c$ be the equation of the straight line.

Then, since the line makes an angle of 60^0 with the axis of x, $m = \tan 60^0 = \sqrt{3}$.

Also, if the point $(\sqrt{3}, 2)$ be on the line, $2 = m \cdot \sqrt{3} + c$, therefore $c = -1$, and the required equation is $y = \sqrt{3}x - 1$.

23. *To find the equation of a straight line drawn through a given point in a given direction.*

Let x', y' be the co-ordinates of the given point, and let the line make with the axis of x an angle $\tan^{-1} m$.

Its equation will then be

$$y = mx + c,$$

and, since (x', y') is on the line,

$$y' = mx' + c,$$

therefore, by subtraction,

$$y - y' = m(x - x') \qquad \text{......(i)}.$$

The line given by (i) passes through the point (x', y') whatever the value of m may be; and by giving a suitable value to m the equation will represent any straight line through the point (x', y').

If then we know that a straight line passes through a particular point (x', y') we at once write down $y - y' = m(x - x')$ for its equation, and find the value of m from the other condition that the line has to satisfy.

24. *To find the equation of a straight line which passes through two given points.*

Take any one of the general forms, for example,

$$y = mx + c \qquad \text{......(i)}.$$

Let the co-ordinates of the two points be x', y' and x'', y'' respectively. Then, since these points are on the line (i), we have

$$y' = mx' + c \qquad \text{......(ii)},$$

and

$$y'' = mx'' + c \qquad \text{......(iii)}.$$

From (i) and (ii), by subtraction,

$$y - y' = m(x - x') \qquad \text{......(iv)}.$$

From (iii) and (ii), by subtraction,

$$y'' - y' = m\,(x'' - x') \ldots\ldots\ldots\ldots\ldots\ldots \text{(v)},$$

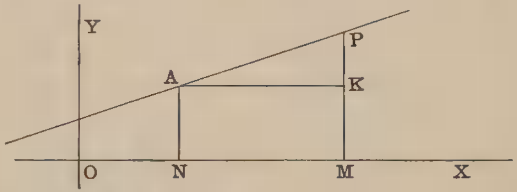
and therefore $$\frac{y - y'}{y'' - y'} = \frac{x - x'}{x'' - x'}.$$

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

This equation could be found at once from a figure.

Ex. The equation of the line joining the points (2, 3) and (3, 1) is $\frac{y-3}{1-3} = \frac{x-2}{3-2}$, or $y + 2x - 7 = 0$.

25. Let the straight line AP make an angle θ with the axis of x. Let the co-ordinates of A be x', y', and those of P be x, y, and let the distance AP be r.



Draw AN, PM parallel to the axis of y, and AK parallel to the axis of x.

Then $AK = AP \cos\theta$, and $KP = AP \sin\theta$,

or $x - x' = r\cos\theta$, and $y - y' = r\sin\theta$.

The equation of the line AP may be written in the form

$$\frac{x - x'}{\cos\theta} = \frac{y - y'}{\sin\theta} = r.$$

26. Let the equation of any straight line be

$$Ax + By + C = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

Let the co-ordinates of any point Q be x', y', and let the line through Q parallel to the axis of y cut the given straight line in the point P whose co-ordinates are x', y''.

Then it is clear from a figure that, so long as Q remains on the same side of the straight line, QP is drawn in the same direction; and that QP is drawn in the opposite direction, if Q be any point whatever on the other side of the straight line.

That is to say, QP is *positive* for all points on one side of the straight line, and *negative* for all points on the other side of the straight line.

Now $$QP = y'' - y' \text{.....................(ii)},$$
and $$Ax' + By' + C = Ax' + By' + C - (Ax' + By'' + C),$$
[for (x', y'') is on the line, and therefore $Ax' + By'' + C = 0$]
$$\therefore\ Ax' + By' + C = -B(y'' - y') \text{.........(iii)}.$$

From (ii) and (iii) we see that $Ax' + By' + C$ is positive for all points on one side of the straight line, and negative for all points on the other side of the line.

If the equation of a straight line be $Ax + By + C = 0$, and the co-ordinates x', y' of any point be substituted in the expression $Ax + By + C$; then if $Ax' + By' + C$ be *positive*, the point (x', y') is said to be on the *positive side* of the line, and if $Ax' + By' + C$ be negative, (x', y') is said to be on the *negative side* of the line.

If the equation of the line be written
$$-Ax - By - C = 0,$$
it is clear that the side which we previously called the *positive side* we should now call the *negative side*.

Ex. 1. The point (3, 2) is on the negative side of $2x - 3y - 1 = 0$, and on the positive side of $3x - 2y - 1 = 0$.

Ex. 2. The points (2, −1) and (1, 1) are on opposite sides of the line $3x + 4y - 6 = 0$.

Ex. 3. Shew that the four points (0, 0), (−1, 1), $(-\frac{7}{12}, 0)$ and $(2, \frac{49}{30})$ are in the four different compartments made by the two straight lines $2x - 3y + 1 = 0$, and $3x - 5y + 2 = 0$.

27. *To find the co-ordinates of the point of intersection of two given straight lines.*

Let the equations of the lines be
$$ax + by + c = 0 \text{ .... (i)},$$
and $$a'x + b'y + c' = 0 \text{ ....(ii)}.$$

Then the co-ordinates of the point which is common to both straight lines will satisfy both equations (i) and (ii).

We have therefore only to find the values of x and y which satisfy both (i) and (ii).

These are given by

$$\frac{x}{bc' - b'c} = \frac{y}{ca' - c'a} = \frac{1}{ab' - a'b}.$$

28. *To find the condition that three straight lines may meet in a point.*

Let the equations of the three straight lines be

$$ax + by + c = 0 \ldots (1), \quad a'x + b'y + c' = 0 \ldots (2),$$
$$a''x + b''y + c'' = 0 \ldots (3).$$

The three straight lines will meet in a point if the point of intersection of two of the lines is on the third.

The co-ordinates of the point of intersection of (1) and (2) are given by

$$\frac{x}{bc' - b'c} = \frac{y}{ca' - c'a} = \frac{1}{ab' - a'b}.$$

The condition that this point may be on (3) is

$$a'' \frac{bc' - b'c}{ab' - a'b} + b'' \frac{ca' - c'a}{ab' - a'b} + c'' = 0,$$

or, $$a''(bc' - b'c) + b''(ca' - c'a) + c''(ab' - a'b) = 0.$$

$y = mx + c$

Rand.

$\frac{y}{b} = 1$

EXAMPLES.

1. Draw the straight lines whose equations are

(i) $x + y = 2$, (ii) $3x - 4y = 12$,
(iii) $4x - 3y + 1 = 0$, and (iv) $2x + 5y + 7 = 0$.

2. Find the equations of the straight lines joining the following pairs of points—(i) (2, 3) and (−4, 1), (ii) (a, b) and (b, a).

Ans. (i) $x - 3y + 7 = 0$, (ii) $x + y = a + b$.

3. Write down the equations of the straight lines which pass through the point (1, −1), and make angles of 150^0 and 30^0 respectively with the axis of x.

Ans. $y + 1 = \mp \frac{1}{\sqrt{3}}(x - 1)$.

4. Write the following equations in the form $x\cos\alpha + y\sin\alpha - p = 0$,—(i) $3x + 4y - 15 = 0$, (ii) $12x - 5y + 10 = 0$.

Ans. (i) $\frac{3}{5}x + \frac{4}{5}y - 3 = 0$, (ii) $-\frac{12}{13}x + \frac{5}{13}y - \frac{10}{13} = 0$.

5. Find the equation of the straight line through (4, 5) parallel to $2x - 3y - 5 = 0$. *Ans.* $2x - 3y + 7 = 0$.

6. Find the equation of the line through (2, 1) parallel to the line joining (2, 3) and (3, −1). *Ans.* $4x + y = 9$.

7. Find the equation of the line through the point (5, 6) which makes equal intercepts on the axes. *Ans.* $x + y = 11$.

8. Find the points of intersection of the following pairs of straight lines (i) $5x + 7y = 99$ and $3x + 2y + 77 = 0$, (ii) $2x - 5y + 1 = 0$ and $x + y + 4 = 0$, (iii) $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$.

Ans. (i) $(-67, 62)$, (ii) $(-3, -1)$, (iii) $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$.

9. Shew that the three lines $5x + 3y - 7 = 0$, $3x - 4y - 10 = 0$, and $x + 2y = 0$ meet in a point.

10. Shew that the three points (0, 11), (2, 3) and (3, −1) are on a straight line.

Also the three points $(3a, 0)$, $(0, 3b)$ and $(a, 2b)$.

11. Find the equations of the sides of the triangle the co-ordinates of whose angular points are (1, 2), (2, 3) and (−3, −5).

Ans. $8x - 5y - 1 = 0$, $7x - 4y + 1 = 0$, $x - y + 1 = 0$.

12. Find the equations of the straight lines each of which passes through one of the angular points and the middle point of the opposite side of the triangle in Ex. 11.

Ans. $2x - y = 0$, $3x - 2y = 0$, $5x - 3y = 0$.

13. Find the equations of the diagonals of the parallelogram the equations of whose sides are $x - a = 0$, $x - b = 0$, $y - c = 0$ and $y - d = 0$.

Ans. $(d - c)x + (a - b)y + bc - ad = 0$ and $(d - c)x + (b - a)y + ac - bd = 0$.

14. What must be the value of a in order that the three lines $3x + y - 2 = 0$, $ax + 2y - 3 = 0$, and $2x - y - 3 = 0$ may meet in a point?

Ans. $a = 5$.

15. In what ratio is the line joining the points (1, 2) and (4, 3) divided by the line joining (2, 3) and (4, 1)? *Ans. The line is bisected.*

16. Are the points (2, 3) and (3, 2) on the same or on opposite sides of the straight line $5y - 6x + 4 = 0$?

17. Shew that the points (0, 0) and (3, 4) are on opposite sides of the line $y-2x+1=0$.

18. Shew that the origin is within the triangle the equations of whose sides are $x-7y+25=0$, $5x+3y+11=0$, and $3x-2y-1=0$.

29. *To find the angle between two straight lines whose equations are given.*

(i) If the equations of the given lines be

$$x\cos\alpha+y\sin\alpha-p=0, \text{ and } x\cos\alpha'+y\sin\alpha'-p'=0,$$

the required angle will be $\alpha-\alpha'$ or $\pi-\alpha-\alpha'$.

For α and α' are the angles which the perpendiculars from the origin on the two lines respectively make with the axis of x, and the angle between any two lines is equal or supplementary to the angle between two lines perpendicular to them.

(ii) If the equations of the lines be

$$y=mx+c, \text{ and } y=m'x+c';$$

then, if θ, θ' be the angles the lines make with the axis of x, $\tan\theta=m$ and $\tan\theta'=m'$;

$$\therefore \tan(\theta-\theta')=\frac{m-m'}{1+mm'};$$

$$\therefore \text{ the required angle is } \tan^{-1}\left(\frac{m-m'}{1+mm'}\right).$$

The lines are perpendicular to one another when $1+mm'=0$, and parallel when $m=m'$.

(iii) If the equations of the lines be

$$ax+by+c=0, \text{ and } a'x+b'y+c'=0,$$

these equations may be written in the forms

$$y=-\frac{a}{b}x-\frac{c}{b}, \text{ and } y=-\frac{a'}{b'}x-\frac{c'}{b'}.$$

Therefore, by (ii), the required angle is

$$\tan^{-1}\frac{-\dfrac{a}{b}+\dfrac{a'}{b'}}{1+\dfrac{aa'}{bb'}}, \text{ or } \tan^{-1}\frac{ba'-b'a}{aa'+bb'}.$$

tan φ = 0 lines are ~~[illegible]~~ parallel

[illegible]

The lines $ax+by+c=0$ and $a'x+b'y+c'=0$ will be at right angles to one another, if $aa'+bb'=0$, and will be parallel to one another if $ba'-b'a=0$, or if $\frac{a}{a'}=\frac{b}{b'}$.

30. The condition of perpendicularity is clearly satisfied by the two lines whose equations are

$$ax+by+c=0 \text{ and } bx-ay+c'=0.$$

The condition is also satisfied by the two lines

$$ax+by+c=0 \text{ and } \frac{x}{a}-\frac{y}{b}+c'=0.$$

Hence if, in the equation of a given straight line, we interchange, or invert, the coefficients of x and y, and alter the sign of one of them, we shall obtain the equation of a perpendicular straight line; and if this line has to satisfy some other condition we must give a suitable value to the constant term.

Ex. 1. The line through the origin perpendicular to $4y+2x=7$ is $2y-4x=0$.

Ex. 2. The line through the point (4, 5) perpendicular to $3x-2y+5=0$ is $2(x-4)+3(y-5)=0$, for it is perpendicular to the given line, and it passes through the point (4, 5).

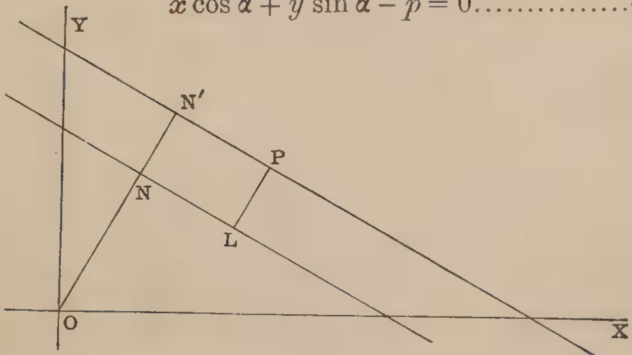
Ex. 3. The acute angle between the lines

$$2x+3y+1=0, \text{ and } x-y=0 \text{ is } \tan^{-1}5.$$

31. *To find the perpendicular distance of a given point from a given straight line.*

Let the equation of the straight line be

$$x\cos\alpha+y\sin\alpha-p=0 \text{..............(i)},$$



and let x', y' be the co-ordinates of the given point P.

The equation

$$x \cos \alpha + y \sin \alpha - p' = 0 \text{................(ii)}$$

is the equation of a straight line parallel to (i).

It will pass through the point (x', y') if

$$x' \cos \alpha + y' \sin \alpha - p' = 0 \text{...............(iii).}$$

Now if PL be the perpendicular from P on the line (i), and ON, ON' the perpendiculars from the origin on the lines (i) and (ii) respectively, then will

$$\begin{aligned} LP &= NN' \\ &= p' - p \\ &= x' \cos \alpha + y' \sin \alpha - p \text{ [from (iii)].} \end{aligned}$$

Hence the length of the perpendicular from any point on the line $x \cos \alpha + y \sin \alpha - p = 0$ is obtained by substituting the co-ordinates of the point in the expression

$$x \cos \alpha + y \sin \alpha - p.$$

The expression $x' \cos \alpha + y' \sin \alpha - p$ is positive so long as p' is positive and greater than p, that is so long as $P(x', y')$ and the origin are on opposite sides of the line.

If the equation of the line be $ax + by + c = 0$, it may be written

$$\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y + \frac{c}{\sqrt{a^2+b^2}} = 0 \text{.........(iv)}$$

which is of the same form as (i) [Art. 20]; therefore the length of the perpendicular from (x', y') on the line is

$$\frac{a}{\sqrt{a^2+b^2}}x' + \frac{b}{\sqrt{a^2+b^2}}y' + \frac{c}{\sqrt{a^2+b^2}},$$

or

$$\frac{ax' + by' + c}{\sqrt{a^2+b^2}} \text{........................(v).}$$

Hence, when the equation of a straight line is given in the form $ax + by + c = 0$, the perpendicular distance of a given point from it is obtained by substituting the co-ordinates of the point in the expression $ax + by + c$, and dividing by the square root of the sum of the squares of the coefficients of x and y.

If the denominator of (v) be always supposed to be positive, the length of the perpendicular from any point on the positive side of the line will be positive, and the length of the perpendicular from any point on the negative side of the line will be negative. [See Art. 26.]

32. *To find the equations of the lines which bisect the angles between two given straight lines.*

The perpendiculars on two straight lines, drawn from any point on either of the lines bisecting the angles between them, will clearly be equal to one another in magnitude.

Hence, if the equations of the lines be

$$ax+by+c=0 \text{.......................(i)},$$

and

$$a'x+b'y+c'=0 \text{.......................(ii)},$$

and (x', y') be any point on either of the bisectors,

$$\frac{ax'+by'+c}{\sqrt{a^2+b^2}} \text{ and } \frac{a'x'+b'y'+c'}{\sqrt{a'^2+b'^2}}$$

must be equal in magnitude.

Hence the point (x', y') is on one or other of the straight lines

$$\frac{ax+by+c}{\sqrt{a^2+b^2}}=\pm\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}} \text{.........(iii)}.$$

The two lines given by (iii) are therefore the required bisectors.

We can distinguish between the two bisectors; for, if we take the denominators to be both positive, and if the upper sign be taken in (iii), $ax+by+c$ and $a'x+b'y+c'$ must *both* be positive or *both* be negative.

Hence in $$\frac{ax+by+c}{\sqrt{a^2+b^2}}=+\frac{a'x+b'x+c'}{\sqrt{a'^2+b'^2}} \text{.........(iv)},$$

every point is on the positive side of *both* the lines (i) and (ii), or on the negative side of *both*.

If the equations are so written that the constant terms are both positive, the origin is on the positive side of both lines; hence (iv) is the bisector of that angle in which the origin lies.

Ex. The bisectors of the angles between the lines $4x-3y+1=0$, and $12x+5y+13=0$ are given by $\frac{4x-3y+1}{5}=\pm\frac{12x+5y+13}{13}$; and the upper sign gives the bisector of the angle in which the origin lies.

33. *To find the equation of a straight line through the point of intersection of two given straight lines.*

The most obvious method of obtaining the required equation is to find the co-ordinates x', y' of the point of intersection of the given lines, and then use the form $y-y'=m(x-x')$ for the equation of any straight line through the point (x', y'). The following method is however sometimes preferable.

Let the equations of the two given straight lines be

$$ax+by+c=0 \quad\text{..................(i)},$$

$$a'x+b'y+c'=0 \quad\text{..................(ii)}.$$

Consider the equation

λ = ?

$$ax+by+c+\lambda(a'x+b'y+c')=0 \quad\text{.........(iii)}.$$

It is the equation of a straight line, since it is of the first degree; and if (x', y') be the point which is common to the two given lines, we shall have

$$ax'+by'+c=0$$

and $$a'x'+b'y'+c'=0,$$

and therefore $$(ax'+by'+c)+\lambda(a'x'+b'y'+c')=0.$$

This last equation shews that the point (x', y') is also on the line (iii).

Hence (iii) is the equation of a straight line passing through the point of intersection of the given lines. Also by giving a suitable value to λ the equation may be made to satisfy any other condition, it may for example be made to pass through any other given point. The equation (iii) therefore represents, for different values of λ, all straight lines through the point of intersection of (i) and (ii).

Ex. Find the equation of the line joining the origin to the point of intersection of $2x+5y-4=0$ and $3x-2y+2=0$.

Any line through the intersection is given by

$$2x+5y-4+\lambda(3x-2y+2)=0.$$

This will pass through (0, 0) if $-4+2\lambda=0$, or if $\lambda=2$;

$$\therefore\ 2x+5y-4+2(3x-2y+2)=0,$$

or $8x+y=0$, is the required equation.

34. If the equations of three straight lines be

$$ax+by+c=0,\ a'x+b'y+c'=0, \text{ and } a''x+b''y+c''=0$$

respectively; and if we can find three constants λ, μ, ν such that the relation

$$\lambda(ax+by+c)+\mu(a'x+b'y+c')+\nu(a''x+b''y+c'')=0 \ldots \text{(i)}$$

is *identically* true, that is to say is true for all values of x and y, then the three straight lines will meet in a point. For if the co-ordinates of any point satisfy any two of the equations of the lines, the relation (i) shews that it will also satisfy the third equation. This principle is of frequent use.

Ex. The three straight lines joining the angular points of a triangle to the middle points of the opposite sides meet in a point.

Let the angular points A, B, C be (x', y'), (x'', y''), (x''', y'''), respectively. Then D, E, F, the middle points of BC, CA, AB respectively, will be

$$\left(\frac{x''+x'''}{2},\ \frac{y''+y'''}{2}\right),\ \left(\frac{x'''+x'}{2},\ \frac{y'''+y'}{2}\right) \text{ and } \left(\frac{x'+x''}{2},\ \frac{y'+y''}{2}\right).$$

The equation of AD will therefore be

$$\frac{y-y'}{\dfrac{y''+y'''}{2}-y'}=\frac{x-x'}{\dfrac{x''+x'''}{2}-x'}$$

or $y(x''+x'''-2x')-x(y''+y'''-2y')+x'(y''+y''')-y'(x''+x''')=0$.

So the equation of BE and CF will be respectively

$$y(x'''+x'-2x'')-x(y'''+y'-2y'')+x''(y'''+y')-y''(x'''+x')=0$$

and $y(x'+x''-2x''')-x(y'+y''-2y''')+x'''(y'+y'')-y'''(x'+x'')=0$.

And, since the three equations when added together vanish identically, the three lines represented by them must meet in a point.

EXAMPLES.

1. Find the angles between the following pairs of straight lines—

(i) $y=2x+5$ and $3x+y=7$,

(ii) $x+2y-4=0$ and $2x-y+1=0$,

(iii) $Ax+By+C=0$ and $(A+B)x-(A-B)y=0$.

Ans. (i) 45^0, (ii) 90^0, (iii) 45^0.

2. Find the equation of the straight line which is perpendicular to $2x+7y-5=0$ and which passes through the point (3, 1).

Ans. $7x-2y=19$.

3. Write down the equations of the lines through the origin perpendicular to the lines $3x+2y-5=0$ and $4x+3y-7=0$. Find the co-ordinates of the points where these perpendiculars meet the lines, and shew that the equation of the line joining these points is $23x+11y-35=0$.

4. Find the perpendicular distances of the point (2, 3) from the lines $4x+3y-7=0$, $5x+12y-20=0$, and $3x+4y-8=0$. *Ans.* 2.

5. Write down the equations of the lines through (1, 1) and $(-2, -1)$ parallel to $3x+4y+7=0$; and find the distance between these lines.

Ans. $\frac{17}{5}$.

6. Find the equations of the two straight lines through the point (2, 3) which make an angle of 45^0 with $x+2y=0$.

Ans. $x-3y+7=0$, $3x+y=9$.

7. Find the equations of the two straight lines which are parallel to $x+7y+2=0$ and at unit distance from the point $(1, -1)$.

Ans. $x+7y+6\pm5\sqrt{2}=0$.

8. Find the equation of the line joining the origin to the point of intersection of the lines $x-4y-7=0$ and $y+2x-1=0$.

Ans. $13x+11y=0$.

9. Find the equation of the straight line joining (1, 1) to the point of intersection of the lines $3x+4y-2=0$ and $x-2y+5=0$. $+16y-1$[illegible])

Ans. $7x+26y-33=0$.

10. Find the equation of the line drawn through the point of intersection of $y-4x-1=0$ and $2x+5y-6=0$, perpendicular to $3y+4x=0$.

Ans. $88y-66x-101=0$.

11. Find the lengths of the perpendiculars drawn from the origin on the sides of the triangle the co-ordinates of whose angular points are (2, 1), (3, 2) and $(-1, -1)$.

12. Find the equations of the straight lines bisecting the angles between the straight lines $4y+3x-12=0$ and $3y+4x-24=0$; and draw a figure representing the four straight lines.

Ans. $y-x+12=0$, $7y+7x-36=0$.

13. Find the equations of the diagonals of the rectangle formed by the lines $x+3y-10=0$, $x+3y-20=0$, $3x-y+5=0$, and $3x-y-5=0$; and shew that they intersect in the point $(\frac{3}{2}, \frac{9}{2})$.

14. Find the area of the triangle formed by the lines $y-x=0$, $y+x=0$, $x-c=0$. *Ans.* c^2.

15. The area of the triangle formed by the straight lines whose equations are $y-2x=0$, $y-3x=0$, and $y=5x+4$ is $\frac{4}{3}$.

16. Find the area of the triangle formed by the lines $y=2x+4$, $2y+3x=5$, and $y+x+1=0$. *Ans.* $\frac{338}{21}$.

17. Shew that the area of the triangle formed by the lines whose equations are $y=m_1x+c_1$, $y=m_2x+c_2$, and $x=0$ is

$$\frac{1}{2}\frac{(c_1-c_2)^2}{m_2-m_1}.$$

18. Shew that the area of the triangle formed by the straight lines whose equations are $y=m_1x+c_1$, $y=m_2x+c_2$, and $y=m_3x+c_3$ is

$$\frac{1}{2}\frac{(c_2-c_3)^2}{m_2-m_3}+\frac{1}{2}\frac{(c_3-c_1)^2}{m_3-m_1}+\frac{1}{2}\frac{(c_1-c_2)^2}{m_1-m_2}.$$ [Use Ex. 17.]

19. Shew that the locus of a point which moves so that the sum of the perpendiculars let fall from it upon two given straight lines is constant is a straight line.

35. *A homogeneous equation of the nth degree will represent n straight lines through the origin.*

Let the equation be

$$Ay^n + By^{n-1}x + Cy^{n-2}x^2 + \ldots + Kx^n = 0 \ldots \text{(i)}.$$

Divide by x^n and we get

$$A\left(\frac{y}{x}\right)^n + B\left(\frac{y}{x}\right)^{n-1} + C\left(\frac{y}{x}\right)^{n-2} + \ldots + K = 0 \ldots \text{(ii)}.$$

Let m_1, m_2, $m_3 \ldots\ldots m_n$ be the roots of this equation. Then it is the same as

$$A\left(\frac{y}{x}-m_1\right)\left(\frac{y}{x}-m_2\right)\left(\frac{y}{x}-m_3\right)\ldots\ldots\left(\frac{y}{x}-m_n\right)=0,$$

and therefore is satisfied when

$$\frac{y}{x}-m_1=0, \text{ when } \frac{y}{x}-m_2=0, \text{\&c.},$$

and in no other cases.

Therefore all the points on the locus represented by (i) are on one or other of the n straight lines

$$y-m_1x=0,\ y-m_2x=0, \ldots\ldots y-m_nx=0.$$

36. *To find the angle between the two straight lines represented by the equation* $Ax^2 + 2Bxy + Cy^2 = 0$.

If the lines be $y - m_1x = 0$, and $y - m_2x = 0$, then $(y - m_1x)(y - m_2x) = 0$ is the same as the given equation

$$y^2 + \frac{2B}{C}xy + \frac{A}{C}x^2 = 0;$$

$$\therefore\ m_1 + m_2 = -\frac{2B}{C} \ldots\ldots (\text{i}) \text{ and } m_1m_2 = \frac{A}{C} \ldots\ldots (\text{ii}).$$

If θ be the angle between the lines,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{2\sqrt{(B^2 - AC)}}{A + C}, \text{ from (i) and (ii).}$$

If $B^2 - AC$ is positive the lines are real, being coincident if $B^2 - AC = 0$.

If $B^2 - AC$ is negative the lines are imaginary, but pass through the real point (0, 0).

The lines given by the equation $Ax^2 + 2Bxy + Cy^2 = 0$, will be at right angles to one another if $A + C = 0$; that is, if the *sum of the coefficients of* x^2 *and* y^2 *is zero.*

37. *To find the condition that the general equation of the second degree may represent two straight lines.*

The most general form of the equation of the second degree is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots (\text{i}).$$

If this is identically equivalent to

$$(lx + my + n)(l'x + m'y + n') = 0 \ldots\ldots\ldots (\text{ii}),$$

we have, by equating coefficients in (i) and (ii),

$$ll' = a, \quad mm' = b, \quad nn' = c,$$

$$mn' + m'n = 2f, \quad nl' + n'l = 2g, \quad lm' + l'm = 2h.$$

By continued multiplication of the last three, we have

$$\begin{aligned} 8fgh &= 2ll'mm'nn' + ll'(m'^2n^2 + m^2n'^2) \\ &\qquad + mm'(n'^2l^2 + n^2l'^2) + nn'(l'^2m^2 + l^2m'^2) \\ &= 2abc + a(4f^2 - 2bc) \\ &\qquad + b(4g^2 - 2ca) + c(4h^2 - 2ab). \end{aligned}$$

Hence $$abc - af^2 - bg^2 - ch^2 + 2fgh = 0 \ldots\ldots\ldots(\text{iii})$$
is the required condition.

Unless the coefficients of x^2 and y^2 are both zero, we can obtain the above result more simply by solving the equation as a quadratic in x or in y.

Suppose a is not zero; then if we solve the quadratic in x, we have

$$ax + hy + g = \pm\sqrt{\{(h^2 - ab)\, y^2 + 2\,(hg - af)\, y + g^2 - ac\}}.$$

Now in order that this may be capable of being reduced to the form $ax + By + C = 0$, it is necessary and sufficient that the quantity under the radical should be a perfect square. The condition for this is

$$(h^2 - ab)\,(g^2 - ac) = (hg - af)^2,$$

which is equivalent to (iii).

38. *To find the equation of the lines joining the origin to the common points of*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots\ldots(\text{i}),$$

and $$lx + my = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii}).$$

Make equation (i) homogeneous and of the second degree by means of (ii), and we get

$$ax^2 + 2hxy + by^2 + 2\,(gx + fy)\,(lx + my) + c\,(lx + my)^2 = 0 \quad \ldots(\text{iii}),$$

which is the equation required.

For equation (iii) being homogeneous represents straight lines through the origin [Art. 35]. To find where the lines (iii) are cut by the line (ii), we must put $lx + my = 1$ in (iii), and we then have the relation (i) satisfied; which shews that the lines (iii) pass through the points common to (i) and (ii).

*39. *To find the equation of the straight lines bisecting the angles between the two straight lines*

$$ax^2 + 2hxy + by^2 = 0.$$

If the given lines make angles θ_1 and θ_2 with the axis of x, then $(y - x\tan\theta_1)\,(y - x\tan\theta_2) = 0$ is the same as the given equation: and we obtain

$$\tan\theta_1 + \tan\theta_2 = -\frac{2h}{b} \dots\dots\dots\dots\dots (\text{i}),$$

and
$$\tan\theta_1 \tan\theta_2 = \frac{a}{b} \dots\dots\dots\dots\dots (\text{ii}).$$

If θ be the angle that one of the bisectors makes with the axis of x, then will

$$\theta = \frac{\theta_1 + \theta_2}{2}, \text{ or } \theta = \frac{\theta_1 + \theta_2}{2} + \frac{\pi}{2};$$

and in either case

$$\tan 2\theta = \tan(\theta_1 + \theta_2),$$

or
$$\frac{2\tan\theta}{1 - \tan^2\theta} = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}.$$

If (x, y) be any point on a bisector, $\frac{y}{x} = \tan\theta$;

hence
$$\frac{2\frac{y}{x}}{1 - \frac{y^2}{x^2}} = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2};$$

therefore, making use of (i) and (ii), we have for the required equation

$$\frac{2xy}{x^2 - y^2} = \frac{2h}{a - b},$$

or
$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}.$$

EXAMPLES.

1. Shew that the two straight lines $y^2 - 2xy\sec\theta + x^2 = 0$ make an angle θ with one another.

2. Shew that the equation $x^2 + xy - 6y^2 + 7x + 31y - 18 = 0$ represents two straight lines, and find the angle between them. *Ans.* 45°.

3. Shew that each of the following equations represents a pair of straight lines, and find the angle between each pair:

(i) $(x - a)(y - a) = 0$, (ii) $x^2 - 4y^2 = 0$,
(iii) $xy = 0$, (iv) $xy - 2x - 3y + 6 = 0$,
(v) $x^2 - 5xy + 4y^2 = 0$, (vi) $x^2 - 5xy + 4y^2 + 3x - 4 = 0$,
(vii) $x^2 + 2xy\cot 2a - y^2 = 0$.

4. For what value of λ does the equation

$$12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0$$

represent two straight lines? Shew that if the equation represents straight lines, the angle between them is $\tan^{-1}\frac{1}{7}$. *Ans.* $\lambda = 2$.

5. For what value of λ does the equation

$$12x^2 + \lambda xy + 2y^2 + 11x - 5y + 2 = 0$$

represent two straight lines? *Ans.* -10, or $-\frac{35}{2}$.

6. For what value of λ does the equation

$$12x^2 + 36xy + \lambda y^2 + 6x + 6y + 3 = 0$$

represent two straight lines? Are the lines real or imaginary? *Ans.* 28.

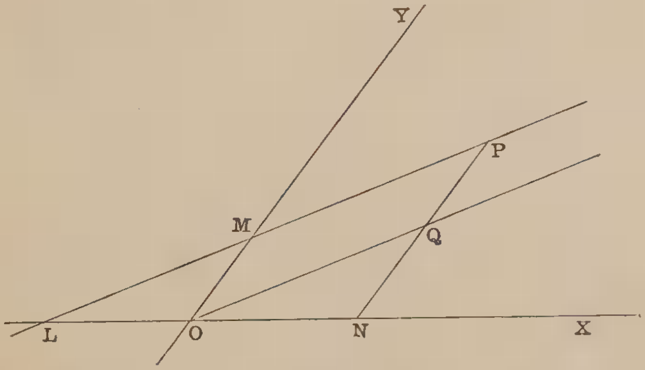
7. For what value of λ does the equation $\lambda xy + 5x + 3y + 2 = 0$ represent two straight lines? *Ans.* $\lambda = \frac{15}{2}$.

8. Shew that the lines joining the origin to the points common to $3x^2 + 5xy - 3y^2 + 2x + 3y = 0$ and $3x - 2y = 1$ are at right angles.

The lines are $3x^2 + 5xy - 3y^2 + (2x + 3y)(3x - 2y) = 0$.

OBLIQUE AXES.

40. *To find the equation of a straight line referred to axes inclined at an angle ω.*



Let LMP be any straight line meeting the axes in the points L, M.

Let x, y be the co-ordinates of any point P on the line.

Draw PN parallel to the axis of y and OQ parallel to the line LMP, as in the figure.

Then $$NP = NQ + QP \quad\text{..................(i)}$$

But $$\frac{NQ}{ON} = \frac{\sin NOQ}{\sin(\omega - NOQ)} = \text{constant} = m \text{ suppose},$$

and $$QP = OM = c \text{ suppose;}$$

therefore (i) becomes $y = mx + c$, which is the required equation.

If θ be the angle which the line makes with the axis of x, then

$$m = \frac{\sin\theta}{\sin(\omega - \theta)},$$

$$\therefore \tan\theta = \frac{m\sin\omega}{1 + m\cos\omega}.$$

41. Many of the investigations in the preceding Articles apply equally whether the axes are rectangular or oblique. These may be easily recognised.

*42. *To find the angle between two straight lines whose equations, referred to axes inclined at an angle ω, are given.*

If the equations of the lines be

$$y = mx + c, \text{ and } y = m'x + c',$$

and if θ, θ' be the angles they make respectively with the axis of x, then [Art. 40]

$$\tan\theta = \frac{m\sin\omega}{1 + m\cos\omega}, \text{ and } \tan\theta' = \frac{m'\sin\omega}{1 + m'\cos\omega};$$

therefore $$\tan(\theta - \theta') = \frac{(m - m')\sin\omega}{1 + (m + m')\cos\omega + mm'} \quad\text{...(i)},$$

or the angle between the lines is

$$\tan^{-1}\frac{(m - m')\sin\omega}{1 + (m + m')\cos\omega + mm'}.$$

The lines will be at right angles to one another, if

$$1 + (m + m')\cos\omega + mm' = 0 \quad\text{..............(ii).}$$

If the equations of the two straight lines be

$$ax + by + c = 0, \text{ and } a'x + b'y + c' = 0,$$

and θ be the angle between them, then $m = -\frac{a}{b}$, and

$m' = -\dfrac{a'}{b'}$, and substituting these values in (i) we have,

$$\tan\theta = \frac{(a'b - ab')\sin\omega}{aa' + bb' - (ab' + a'b)\cos\omega}.$$

The lines will be at right angles to one another, if

$$aa' + bb' - (ab' + a'b)\cos\omega = 0 \ldots\ldots\ldots\ldots \text{(iii)}.$$

*43. *To find the perpendicular distance of any point* (f, g) *from the line* $Ax + By + C = 0$.

Let the line cut the axes of x and y in the points K, L respectively, and let P be the point whose co-ordinates are f, g, and let PN be the perpendicular from it on the line LK. Then

$$\triangle PLK = \triangle POK + \triangle PLO - \triangle LOK \ldots\ldots \text{(i)},$$

$$\therefore PN.LK = OK.g\sin\omega + OL.f\sin\omega - OK.OL\sin\omega \ldots \text{(ii)}.$$

The relation expressed in (i) requires to be modified for different positions of the point and of the line, unless we make some convention with respect to the sign of the area of a triangle, but the equation (ii) is universally true. The student should convince himself of the truth of this by drawing different figures.

Now $$OK = -\frac{C}{A},\quad OL = -\frac{C}{B};$$

also $$LK^2 = OK^2 + OL^2 - 2OK.OL\cos\omega$$
$$= \frac{C^2}{A^2B^2}(A^2 + B^2 - 2AB\cos\omega);$$

$\therefore$ from (ii) $$PN = \frac{Af + Bg + C}{\sqrt{\{A^2 + B^2 - 2AB\cos\omega\}}}\sin\omega.$$

*44. *To find the angle between the lines*

$$ax^2 + 2hxy + by^2 = 0,$$

the axes being inclined at an angle ω.

If the lines be $y = m'x$ and $y = m''x$,

then will $$m' + m'' = -\frac{2h}{b},$$

and $$m' m'' = \frac{a}{b};$$

whence $$m' - m'' = \frac{2\sqrt{h^2 - ab}}{b}.$$

But the angle between $y = m'x$ and $y = m''x$ is

$$\tan^{-1} \frac{(m' - m'') \sin \omega}{1 + (m' + m'') \cos \omega + m'm''} \qquad \text{[Art. 42];}$$

therefore the angle required is

$$\tan^{-1} \frac{2\sqrt{\{h^2 - ab\}} \sin \omega}{b - 2h \cos \omega + a}.$$

The lines $ax^2 + 2hxy + by^2 = 0$ are at right angles to one another, if

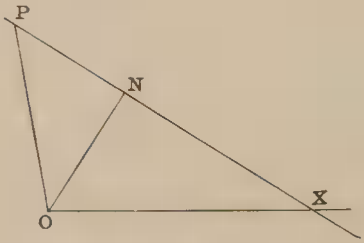
$$a + b - 2h \cos \omega = 0.$$

POLAR CO-ORDINATES.

45. *To find the polar equation of a straight line.*

Let ON be the perpendicular on the given line from the origin, and let $ON = p$, and $XON = \alpha$.

Let P be any point on the line, and let the co-ordinates of P be r, θ.



Then, in the figure, $\angle NOP$ is $(\theta - \alpha)$, and

$$OP \cos NOP = ON.$$

Therefore the required equation is

$$r \cos (\theta - \alpha) = p.$$

This equation may also be obtained by writing $r \cos \theta$ for x, and $r \sin \theta$ for y in the equation $x \cos \alpha + y \sin \alpha = p$.

46. *To find the polar equation of the line through two given points.*

Let P, Q be the given points and let their co-ordinates be r', θ' and r'', θ'' respectively.

Let R be any other point on the line, and let the co-ordinates of R be (r, θ).

Then, since

$$\triangle POQ + \triangle QOR - \triangle POR = 0,$$

we have

$$r'r'' \sin(\theta'' - \theta') + r''r \sin(\theta - \theta'') - rr' \sin(\theta - \theta') = 0.$$

The required equation is therefore,

$$r'r'' \sin(\theta'' - \theta') + r''r \sin(\theta - \theta'') + rr' \sin(\theta' - \theta) = 0.$$

EXAMPLES.

1. Shew that the lines given by the equation $y^2 - x^2 = 0$ are at right angles to one another whatever the angle between the axes may be.

2. Find the equation of the straight line passing through the point (1, 2) and cutting the line $x + 2y = 0$ at right angles, the axes being inclined at an angle of 60^0. *Ans.* $x = 1$.

3. Find the angle the straight line $y = 5x + 6$ makes with the axis of x, the axes being inclined at an angle whose cosine is $\frac{3}{5}$. *Ans.* 45^0.

4. If $y = mx + c$ and $y = m'x + c'$ make equal angles with the axis of x, then will $m + m' + 2mm' \cos\omega = 0$.

5. If the lines $Ax^2 + 2Bxy + Cy^2 = 0$ make equal angles with the axis of x, then will $B = A \cos\omega$.

6. Shew that the lines given by the equation

$$x^2 + 2xy \cos\omega + y^2 \cos 2\omega = 0$$

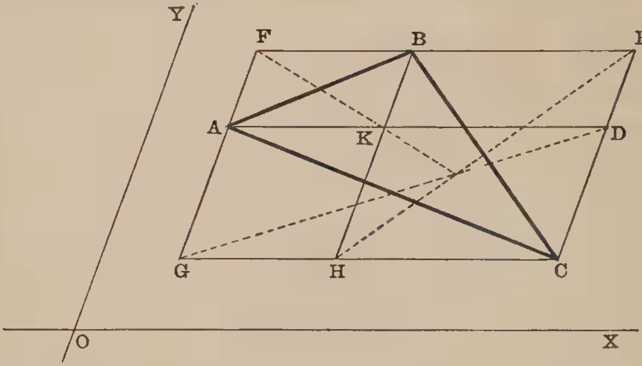
are at right angles to one another, the axes being inclined at an angle ω.

7. Find the polar co-ordinates of the foot of the perpendicular from the pole on the line joining the two points (r_1, θ_1), (r_2, θ_2).

47. We shall conclude this chapter by the solution of some examples.

(1) *On the sides of a triangle as diagonals, parallelograms are described, having their sides parallel to two given straight lines; shew that the other diagonals of these parallelograms will meet in a point.*

Take any two lines parallel to the sides of the parallelograms for the axes. Let A, B, C, the angular points of the triangle, be (x', y'), (x'', y'') and (x''', y''') respectively.



Then the extremities of the other diagonal of the parallelogram of which AB is one diagonal will be seen to be (x', y'') and (x'', y').

Therefore the equation of the diagonal FK will be

$$\frac{y - y''}{y' - y''} = \frac{x - x'}{x'' - x'},$$

or

$$x(y' - y'') + y(x' - x'') + x''y'' - x'y' = 0.$$

Similarly the equation of HE will be

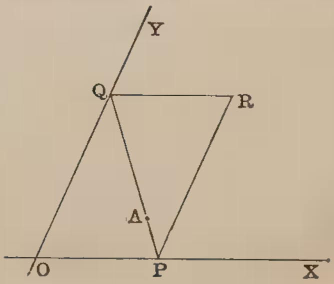
$$x(y'' - y''') + y(x'' - x''') + x'''y''' - x''y'' = 0,$$

and the equation of GD will be

$$x(y''' - y') + y(x''' - x') + x'y' - x'''y''' = 0.$$

The sum of these three equations vanishes identically, therefore the three straight lines meet in a point. [Art. 34.]

(2) *Any straight line is drawn through a fixed point* A *cutting two given straight lines* OX, OY *in the points* P, Q *respectively, and the parallelogram* OPRQ *is completed: find the equation of the locus of* R.



Take the two given lines for the axes, and let the co-ordinates of the point A be f, g.

Let the equation of the line PQ in any one of its possible positions be

$$\frac{x}{\alpha}+\frac{y}{\beta}=1 \quad\text{.................................}\quad \text{(i)}.$$

Then the co-ordinates of the point R will be α and β.

But, since the line PQ passes through the point (f, g), the values $x=f$, $y=g$ satisfy the equation (i). Therefore

$$\frac{f}{\alpha}+\frac{g}{\beta}=1 \quad\text{.................................}\quad \text{(ii)}.$$

Hence the co-ordinates α and β of the point R always satisfy the relation (ii). Calling the co-ordinates of the point R, x and y instead of α and β, we have for the equation of its locus

$$\frac{f}{x}+\frac{g}{y}=1.$$

(3) *Through a fixed point* O *any straight line is drawn meeting two given parallel straight lines in* P *and* Q; *through* P *and* Q *straight lines are drawn in fixed directions, meeting in* R: *prove that the locus of* R *is a straight line.*

Take the fixed point O for origin, and the axis of y parallel to the two parallel straight lines; and let the equations of these parallel lines be $x=a$, $x=b$.

Then, if the equation of OPQ be $y=mx$, the abscissa of P is a, and therefore its ordinate ma; also the abscissa of Q is b, and therefore its ordinate mb.

Let PR be always parallel to $y=m'x$ and QR always parallel to $y=m''x$, then the equation of PR will be

$$y-ma=m'(x-a) \quad\text{.................................}\quad \text{(i)},$$

and the equation of QR will be

$$y-mb=m''(x-b) \quad\text{.................................}\quad \text{(ii)}.$$

At the point R the relations (i) and (ii) will both hold, and we can find, for any particular value of m, the co-ordinates of the point R by solving the simultaneous equations (i) and (ii). This however is not what we want. What we require is the algebraic relation which is satisfied by the co-ordinates (x, y) of the point R, whatever the value of m may be. To find this we have only to eliminate m between the equation (i) and (ii).

The result is

$$(b-a)\,y = m'b\,(x-a) - m''a\,(x-b).$$

This equation is of the first degree, and therefore the required locus is a straight line.

(4) *To find the centres of the inscribed circle and of the escribed circles of a triangle whose angular points are given.*

Let (x', y'), (x'', y''), (x''', y''') be the angular points A, B, C respectively. The equation of BC is

$$y\,(x''-x''') - x\,(y''-y''') + y''x''' - x''y''' = 0 \;\ldots\ldots\ldots\ldots\text{(i)},$$

the equation of CA is

$$y\,(x'''-x') - x\,(y'''-y') + y'''x' - x'''y' = 0 \;\ldots\ldots\ldots\ldots\text{(ii)},$$

and the equation of AB is

$$y\,(x'-x'') - x\,(y'-y'') + y'x'' - x'y'' = 0 \;\ldots\ldots\ldots\ldots\text{(iii)}.$$

The perpendiculars on these lines from the centre of any one of the circles are equal in *magnitude*.

The centres of the four circles are therefore [Art. 31] given by

$$\pm\frac{y\,(x''-x''') - x\,(y''-y''') + y''x''' - x''y'''}{\sqrt{(x''-x''')^2 + (y''-y''')^2}}$$

$$= \pm\frac{y\,(x'''-x') - x\,(y'''-y') + y'''x' - x'''y'}{\sqrt{(x'''-x')^2 + (y'''-y')^2}}$$

$$= \pm\frac{y\,(x'-x'') - x\,(y'-y'') + y'x'' - x'y''}{\sqrt{(x'-x'')^2 + (y'-y'')^2}} \;\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

If the co-ordinates of the angular points A, B, C of the triangle be substituted in the equations (i), (ii), (iii) respectively, the left hand members of all three will be the same. Hence, [Art. 26] the angular points of the triangle are either all on the positive sides of the lines (i), (ii), (iii), or all on the negative sides.

The perpendiculars from the centre of the inscribed circle on the sides of the triangle are *all* drawn in the *same* direction as those from the angular points of the triangle. Hence in (iv) the signs of all the ambiguities are positive for the inscribed circle.

For the escribed circles the signs are $-++$, $+-+$, and $++-$ respectively.

EXAMPLES ON CHAPTER II.

1. A straight line moves so that the sum of the reciprocals of its intercepts on two fixed intersecting lines is constant; shew that it passes through a fixed point.

2. Prove that $bx^2 - 2hxy + ay^2 = 0$ represents two straight lines at right angles respectively to the straight lines

$$ax^2 + 2hxy + by^2 = 0.$$

3. Find the equation to the n straight lines through (a, b) perpendicular respectively to the lines given by the equation

$$p_0y^n + p_1y^{n-1}x + p_2y^{n-2}x^2 + \ldots\ldots + p_nx^n = 0.$$

4. Find the angles between the straight lines represented by the equation

$$x^3 + 3x^2y - 3xy^2 - y^3 = 0.$$

5. OA, OB are two fixed straight lines, A, B being fixed points; P, Q are any two points on these lines such that the ratio of AP to BQ is constant; shew that the locus of the middle point of PQ is a straight line.

6. If a straight line be such that the sum of the perpendiculars upon it from any number of fixed points is zero, shew that it will pass through a fixed point.

7. PM, PN are the perpendiculars from a point P on two fixed straight lines which meet in O; MQ, NQ are drawn parallel to the fixed straight lines to meet in Q; prove that, if the locus of P be a straight line, the locus of Q will also be a straight line.

8. A straight line OPQ is drawn through a fixed point O, meeting two fixed straight lines in the points P, Q, and in the straight line OPQ a point R is taken such that OP, OR, OQ are in harmonic progression; shew that the locus of R is a straight line.

9. Find the equations of the diagonals of the parallelogram formed by the lines

$$\alpha = 0,\ \ \alpha = c,\ \ \alpha' = 0,\ \ \alpha' = c,$$

where $$\alpha \equiv x \cos a + y \sin a - p,$$

and $$\alpha' \equiv x \cos a' + y \sin a' - p'.$$

10. *ABCD* is a parallelogram. Taking *A* as pole, and *AB* as initial line, find the polar equations of the four sides and of the two diagonals.

11. From a given point (h, k) perpendiculars are drawn to the axes and their feet are joined; prove that the length of the perpendicular from (h, k) upon this line is

$$\frac{hk \sin^2 \omega}{\sqrt{\{h^2 + k^2 + 2hk \cos \omega\}}},$$

and that its equation is $hx - ky = h^2 - k^2$.

12. The distance of a point (x_1, y_1) from each of two straight lines, which pass through the origin of co-ordinates, is δ; prove that the two lines are given by

$$(x_1y - xy_1)^2 = \delta^2 (x^2 + y^2).$$

13. Shew that the lines *FC*, *KB*, and *AL* in the figure to Euclid I. 47 meet in a point.

14. Find the equations of the sides of a square the co-ordinates of two opposite angular points of which are 3, 4 and 1, -1.

15. Find the equation of the locus of the vertex of a triangle which has a given base and given difference of base angles.

16. Find the equation of the locus of a point at which two given portions of the same straight line subtend equal angles.

17. The product of the perpendiculars drawn from a point on the lines

$$x \cos \theta + y \sin \theta = a, \quad x \cos \phi + y \sin \phi = a$$

is equal to the square of the perpendicular drawn from the same point on the line

$$x \cos \frac{\theta + \phi}{2} + y \sin \frac{\theta + \phi}{2} = a \cos \frac{\theta - \phi}{2};$$

shew that the equation of the locus of the point is $x^2 + y^2 = a^2$.

18. *PA*, *PB* are straight lines passing through the fixed points *A*, *B* and intercepting a constant length on a given straight line; find the equation of the locus of *P*.

19. The area of the parallelogram formed by the straight lines $3x+4y=7a_1$, $3x+4y=7a_2$, $4x+3y=7b_1$, and $4x+3y=7b_2$ is $7(a_1-a_2)(b_1-b_2)$.

20. Shew that the area of the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ is

$$\frac{n^2\sqrt{(h^2-ab)}}{am^2-2hlm+bl^2}.$$

21. Shew that the angle between one of the lines given by $ax^2+2hxy+by^2=0$, and one of the lines

$$ax^2+2hxy+by^2+\lambda(x^2+y^2)=0,$$

is equal to the angle between the other two lines of the system.

22. Find the condition that one of the lines

$$ax^2+2hxy+by^2=0,$$

may coincide with one of the lines

$$a'x^2+2h'xy+b'y^2=0.$$

23. Find the condition that one of the lines

$$ax^2+2hxy+by^2=0,$$

may be perpendicular to one of the lines

$$a'x^2+2h'xy+b'y^2=0.$$

24. Shew that the point $(1, 8)$ is the centre of the inscribed circle of the triangle the equations of whose sides are

$4y+3x=0$, $12y-5x=0$, $y-15=0$, respectively.

25. Shew that the co-ordinates of the centre of the circle inscribed in the triangle the co-ordinates of whose angular points are $(1, 2)$, $(2, 3)$ and $(3, 1)$ are $\frac{1}{6}(8+\sqrt{10})$ and $\frac{1}{6}(16-\sqrt{10})$. Find also the centres of the escribed circles distinguishing the different cases.

26. If the axes be rectangular, prove that the equation

$$(x^2-3y^2)x=my(y^2-3x^2)$$

represents three straight lines through the origin making equal angles with one another.

27. Shew that the product of the perpendiculars from the point (x', y') on the lines $ax^2+2hxy+by^2=0$, is equal to

$$\frac{ax'^2+2hx'y'+by'^2}{\sqrt{(a-b)^2+4h^2}}.$$

28. If p_1, p_2 be the perpendiculars from (x, y) on the straight lines $ax^2 + 2hxy + by^2 = 0$, prove that

$$(p_1^2 + p_2^2)\{(a-b)^2 + 4h^2\} = 2(a-b)(ax^2 - by^2) \\ + 4h(a+b)xy + 4h^2(x^2+y^2).$$

29. Shew that the locus of a point such that the product of the perpendiculars from it upon the three straight lines represented by

$$ay^3 + by^2x + cyx^2 + dx^3 = 0$$

is constant and equal to k^3

is $$ay^3 + by^2x + cyx^2 + dx^3 - k^3\sqrt{(a-c)^2 + (b-d)^2} = 0.$$

30. Shew that the condition that two of the lines represented by the equation

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 = 0$$

may be at right angles is

$$A^2 + 3AC + 3BD + D^2 = 0.$$

31. Shew that the equation

$$a(x^4 + y^4) - 4bxy(x^2 - y^2) + 6cx^2y^2 = 0$$

represents two pairs of straight lines at right angles, and that, if $2b^2 = a^2 + 3ac$, the two pairs will coincide.

32. The necessary and sufficient condition that two of the lines represented by the equation

$$ay^4 + bxy^3 + cx^2y^2 + dx^3y + ex^4 = 0$$

should be at right angles is

$$(b+d)(ad+be) + (e-a)^2(a+c+e) = 0.$$

33. Shew that the straight lines joining the origin to the points of intersection of the two curves

$$ax^2 + 2hxy + by^2 + 2gx = 0,$$

and $$a'x^2 + 2h'xy + b'y^2 + 2g'x = 0,$$

will be at right angles to one another, if $g'(a+b) = g(a'+b')$.

34. Prove that, if the perpendiculars from the angular points of one triangle upon the sides of a second meet in a point, the perpendiculars from the angular points of the second on the sides of the first will also meet in a point.

35. If the angular points of a triangle lie on three fixed straight lines which meet in a point, and two of the sides pass through fixed points, then will the third side also pass through a fixed point.

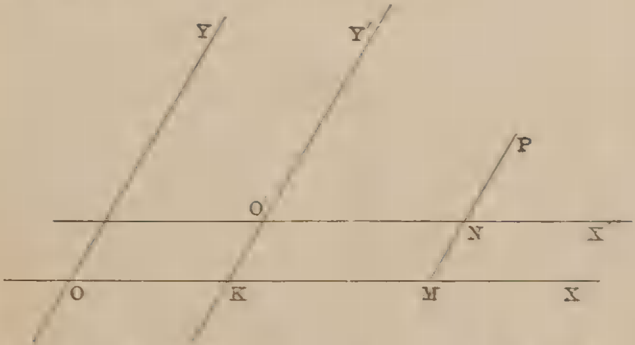
CHAPTER III.

CHANGE OF AXES. ANHARMONIC RATIOS, OR CROSS RATIOS. INVOLUTION.

CHANGE OF AXES.

48. When we know the equation of a curve referred to one set of axes, we can deduce the equation referred to another set of axes.

49. *To change the origin of co-ordinates without changing the direction of the axes.*



Let OX, OY be the original axes; $O'X'$, $O'Y'$ the new axes; $O'X'$ being parallel to OX, and $O'Y'$ being parallel to OY. Let h, k be the co-ordinates of O' referred to the original axes.

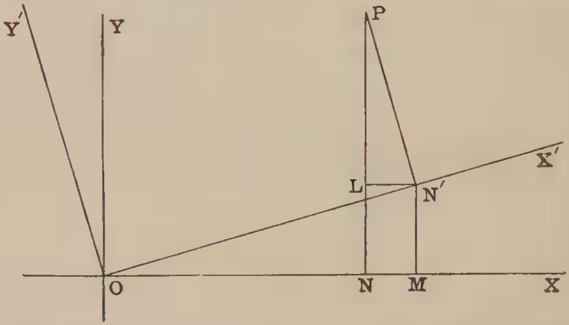
Let P be any point whose co-ordinates referred to the old axes are x, y, and referred to the new axes x', y'. Draw PM parallel to OY, cutting OX in M and $O'X'$ in N.

Then $$x = OM = OK + KM = OK + O'N = h + x',$$ $$y = MP = MN + NP = KO' + NP = k + y'.$$

Hence the old co-ordinates of any point are found in terms of the new co-ordinates; and if these values be substituted in the given equation, the new equation of the curve will be obtained.

In the above the axes may be rectangular or oblique.

50. *To change the direction of the axes without changing the origin, both systems being rectangular.*



Let OX, OY be the original axes; OX', OY' the new axes; and let the angle $XOX' = \theta$.

Let P be any point whose co-ordinates are x, y referred to the original axes, and x', y' referred to the new axes. Draw PN perpendicular to OX, PN' perpendicular to OX', $N'M$ perpendicular to OX, and $N'L$ perpendicular to PN, as in the figure.

Then $$\begin{aligned} x &= ON = OM - NM = OM - LN' \\ &= ON' \cos\theta - N'P \sin\theta \\ &= x' \cos\theta - y' \sin\theta; \\ y &= NP = NL + LP = MN' + LP \\ &= ON' \sin\theta + N'P \cos\theta \\ &= x' \sin\theta + y' \cos\theta. \end{aligned}$$

Hence the old co-ordinates of any point are found in terms of the new co-ordinates; and if these values be substituted in the given equation, the new equation of the curve will be obtained.

Ex. 1. *What does the equation* $3x^2+2xy+3y^2-18x-22y+50=0$ *become when referred to rectangular axes through the point* (2, 3), *the new axis of* x *making an angle of* 45^0 *with the old?*

First change the origin, by putting $x'+2$, $y'+3$ for x, y respectively.

The new equation will be

$$3(x'+2)^2+2(x'+2)(y'+3)+3(y'+3)^2-18(x'+2)-22(y'+3)+50=0;$$

which reduces to $3x'^2+2x'y'+3y'^2-1=0$,

or, suppressing the accents, to

$$3x^2+2xy+3y^2=1 \text{ .... (i).}$$

To turn the axes through an angle of 45^0 we must write $x'\frac{1}{\sqrt{2}}-y'\frac{1}{\sqrt{2}}$ for x, and $x'\frac{1}{\sqrt{2}}+y'\frac{1}{\sqrt{2}}$ for y. Equation (i) will then be

$$3\left(\frac{x'-y'}{\sqrt{2}}\right)^2+2\frac{x'-y'}{\sqrt{2}}\cdot\frac{x'+y'}{\sqrt{2}}+3\left(\frac{x'+y'}{\sqrt{2}}\right)^2=1,$$

which reduces to $4x'^2+2y'^2=1$.

Ex. 2. What does the equation $x^2-y^2+2x+4y=0$ become when the origin is transferred to the point $(-1, 2)$? *Ans.* $x^2-y^2+3=0$.

Ex. 3. Shew that the equation $6x^2+5xy-6y^2-17x+7y+5=0$, when referred to axes through a certain point parallel to the original axes will become $6x^2+5xy-6y^2=0$.

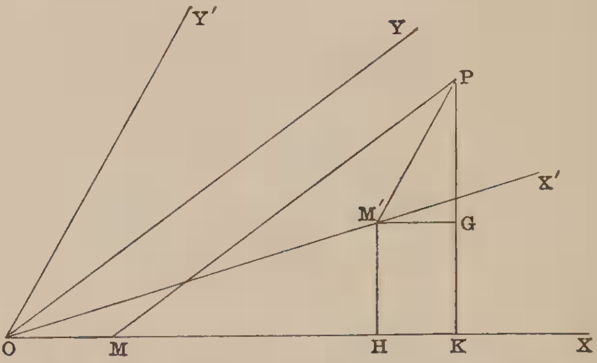
Ex. 4. What does the equation $4x^2+2\sqrt{3}xy+2y^2-1=0$ become when the axes are turned through an angle of 30^0? *Ans.* $5x^2+y^2-1=0$.

Ex. 5. Transform the equation $x^2-2xy+y^2+x-3y=0$ to axes through the point $(-1, 0)$ parallel to the lines bisecting the angles between the original axes. *Ans.* $\sqrt{2}y^2-x=0$.

Ex. 6. Transform the equation $x^2+cxy+y^2=a^2$, by turning the rectangular axes through the angle $\frac{\pi}{4}$.

51. *To change from one set of oblique axes to another, without changing the origin.*

Let OX, OY be the original axes inclined at an angle ω; and OX', OY' be the new axes inclined at an angle ω'; and let the angle $XOX' = \theta$.



Let P be any point whose co-ordinates are x, y referred to the original axes, and x', y' referred to the new axes; so that in the figure $OM = x$, $MP = y$, $OM' = x'$, and $M'P = y'$, MP being parallel to OY and $M'P$ parallel to OY'.

Draw PK and $M'H$ perpendicular to OX, and $M'G$ perpendicular to PK.

Then $$KP = KG + GP = HM' + GP;$$

$$\therefore\ y \sin \omega = x' \sin XOX' + y' \sin XOY'$$
$$= x' \sin \theta + y' \sin (\theta + \omega').$$

Similarly, by drawing PL perpendicular to OY, we can shew that

$$x \sin \omega = x' \sin X'OY - y' \sin YOY'$$
$$= x' \sin (\omega - \theta) - y' \sin (\omega' + \theta - \omega).$$

These formulæ are very rarely used. The results which would be obtained by the change of axes are generally found in an indirect manner, as in the following Article.

*52. *If by any change of axes* $ax^2 + 2hxy + by^2$ *be changed into* $a'x'^2 + 2h'x'y' + b'y'^2$, *then will*

$$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'},$$

and $$\frac{ab-h^2}{\sin^2\omega}=\frac{a'b'-h'^2}{\sin^2\omega'},$$

where ω *and* ω' *are the angles of inclination of the two sets of axes.*

If O be the origin and P be any point whose co-ordinates are x, y referred to the old axes and x', y' referred to the new, then OP^2 is equal to $x^2+y^2+2xy\cos\omega$, and also equal to $x'^2+y'^2+2x'y'\cos\omega'$.

Hence $x^2+y^2+2xy\cos\omega$ is changed into

$$x'^2+y'^2+2x'y'\cos\omega'.$$

Also, by supposition,

$ax^2+2hxy+by^2$ is changed into $a'x'^2+2h'x'y'+b'y'^2$.

Therefore, if λ be any constant,

$ax^2+2hxy+by^2+\lambda(x^2+2xy\cos\omega+y^2)$ will be changed into $a'x'^2+2h'x'y'+b'y'^2+\lambda(x'^2+2x'y'\cos\omega'+y'^2)$.

Therefore, if λ be so chosen that one of these expressions is a perfect square, the other will also be a perfect square for the same value of λ.

The first will be a perfect square if

$$(a+\lambda)(b+\lambda)-(h+\lambda\cos\omega)^2=0,$$

and the second if

$$(a'+\lambda)(b'+\lambda)-(h'+\lambda\cos\omega')^2=0.$$

These two quadratic equations for finding λ must have the same roots. Writing them in the forms

$$\lambda^2+\frac{a+b-2h\cos\omega}{\sin^2\omega}\lambda+\frac{ab-h^2}{\sin^2\omega}=0,$$

and $$\lambda^2+\frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'}\lambda+\frac{a'b'-h'^2}{\sin^2\omega'}=0,$$

we see that

$$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'}\;\ldots\ldots\text{(i)},$$

and $$\frac{ab-h^2}{\sin^2\omega}=\frac{a'b'-h'^2}{\sin^2\omega'}\ \text{.......... ...(ii).}$$

If both sets of axes are at right angles these equations take the simpler forms

$$a+b=a'+b', \text{ and } ab-h^2=a'b'-h'^2 \text{.........(iii).}$$

53. *The degree of an equation is not altered by any alteration of the axes.*

For, from Articles 49, 50, and 51, we see that, however the axes may be changed, the new equation is obtained by substituting for x and y expressions of the form

$$lx'+my'+n, \text{ and } l'x'+m'y'+n'.$$

These expressions are of the first degree, and therefore if they replace x and y in the equation the degree of the equation will not be *raised*. Neither can the degree of the equation be *lowered*, for, if it were, by returning to the original axes, and therefore to the original equation, the degree would be raised.

Ex. 1. Prove, by actual transformation of rectangular axes, that if $ax^2+2hxy+by^2$ become $a'x'^2+2h'x'y'+b'y'^2$, then will $a+b=a'+b'$, and $h^2-ab=h'^2-a'b'$.

Ex. 2. If the formula for transformation from one set of axes to another with the same origin be $x=mx'+ny'$, $y=m'x'+n'y'$; shew that

$$\frac{m^2+m'^2-1}{n^2+n'^2-1}=\frac{mm'}{nn'}.$$

[$x^2+y^2+2xy\cos\omega$ will become $x'^2+y'^2+2x'y'\cos\omega'$. Substitute therefore the given expressions for x and y, and equate coefficients of x'^2 and y'^2 to unity, and then eliminate $\cos\omega$.]

ANHARMONIC OR CROSS RATIOS.

*54. A set of points on a straight line is called a *range*; and a set of straight lines passing through a point is called a *pencil*; each line is called a *ray* of the pencil.

If P, Q, R, S be four points on a straight line, the ratio $\frac{PQ}{RQ} : \frac{PS}{RS}$ or $PQ \,.\, RS : PS \,.\, RQ$ is called the *anharmonic ratio* or *cross ratio* of the range P, Q, R, S, and is expressed by the notation $\{PQRS\}$.

If OP, OQ, OR, OS be a pencil of four straight lines the ratio $\sin POQ \,.\, \sin ROS : \sin POS \,.\, \sin ROQ$ is called the *anharmonic ratio* or *cross ratio* of the pencil, and is expressed by the notation $O\{PQRS\}$.

If the cross ratio of a pencil or of a range is equal to -1 it is said to be *harmonic*.

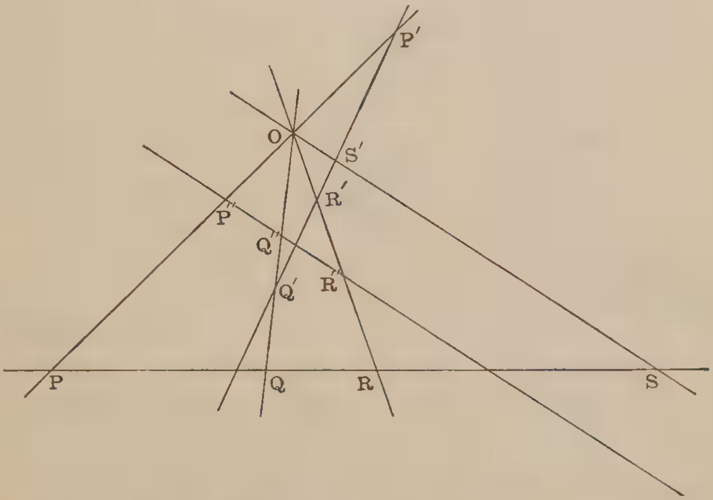
It is easy to shew that if $\{PQRS\} = -1$, then

$$PQ : PS :: PR - PQ : PS - PR,$$

so that PQ, PR, PS are in harmonical progression.

If P, Q, R, S be a harmonic range, then Q and S are said to be *harmonically conjugate* with respect to P and R.

*55. *If four straight lines intersecting in a point* O *be cut by any straight line in the points* P, Q, R, S, *the cross ratio of the range* P, Q, R, S *will be equal to that of the pencil* OP, OQ, OR, OS.



For, if p be the length of the perpendicular from O on the line $PQRS$, we have

$$p \,.\, PQ = OP \,.\, OQ \sin POQ,$$
$$p \,.\, RS = OR \,.\, OS \sin ROS,$$
$$p \,.\, PS = OP \,.\, OS \sin POS,$$
$$p \,.\, RQ = OQ \,.\, OR \sin ROQ.$$

Hence $$\frac{PQ \,.\, RS}{PS \,.\, RQ} = \frac{\sin POQ \,.\, \sin ROS}{\sin POS \,.\, \sin ROQ};$$

that is $$\{PQRS\} = O\{PQRS\}.$$

If the pencil be cut by any two straight lines in the points P, Q, R, S and P', Q', R', S' respectively, as in the figure, the cross ratios of the ranges P, Q, R, S and P', Q', R', S' will be equal to one another, since they are both equal to the cross ratio of the pencil.

If we draw the transversal $P''Q''R''$ parallel to OS, it will meet OS at an infinitely distant point, and, representing this point by the symbol ∞, we have

$$O\{PQRS\} = \{P''Q''R''\infty\} = \frac{P''Q'' \,.\, R''\infty}{P''\infty \,.\, R''Q''} = \frac{P''Q''}{R''Q''};$$

since $\dfrac{R''\infty}{P''\infty}$ is unity.

*56. *To find the cross ratio of the pencil formed by the lines whose equations are*

$$x = 0, \;\; y - mx = 0, \;\; y = 0 \;\; \textit{and} \;\; y - m'x = 0.$$

Draw the line $x = h$ cutting $y - mx = 0$ in P, the axis of x in N, and $y - m'x = 0$ in Q.

Then $NQ = m'h$, and $NP = mh$.

$$O\{YPXQ\} = \{\infty PNQ\} = \frac{\infty P \,.\, NQ}{\infty Q \,.\, NP} = \frac{NQ}{NP} = \frac{m'h}{mh} = \frac{m'}{m}.$$

From the above we see that the four lines

$$x = 0, \;\; y - mx = 0, \;\; y = 0 \;\text{ and }\; y + mx = 0$$

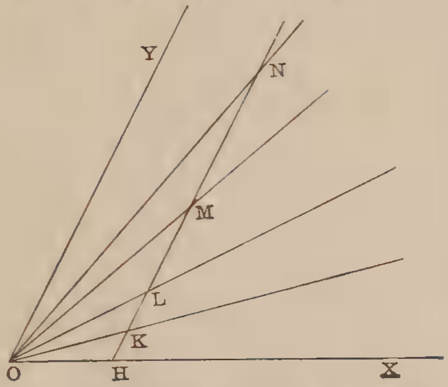
form a harmonic pencil.

If the axes are at right angles to one another the lines $y - mx = 0$, and $y + mx = 0$ make equal angles with either axis.

Hence, if a pencil be harmonic and two alternate rays be at right angles to one another, they will bisect the internal and external angles between the other two.

*57. *To find the cross ratio of the pencil formed by the four lines*

$$y = kx,\ y = lx,\ y = mx,\ y = nx.$$



Draw any line parallel to OY cutting the given lines in the points K, L, M, N respectively, and the axis of x in H, and let $OH = x'$.

Then $$O\{KLMN\} = \frac{KL \,.\, MN}{KN \,.\, ML}.$$

Now $$\begin{aligned} KL &= HL - HK = lx' - kx', \\ MN &= HN - HM = nx' - mx', \\ KN &= HN - HK = nx' - kx', \\ ML &= HL - HM = lx' - mx'. \end{aligned}$$

Hence $$O\{KLMN\} = \frac{(k-l)(m-n)}{(k-n)(m-l)}.$$

*58. *To find the condition that the lines given by the two equations* $ax^2 + 2hxy + by^2 = 0$ *and* $a'x^2 + 2h'xy + b'y^2 = 0$ *may be harmonically conjugate.*

Let the pairs of lines be $y = \alpha x,\ y = \alpha' x$;

and $$y = \beta x,\ y = \beta' x.$$

Then, if $y = \alpha x,\ y = \beta x,\ y = \alpha' x$, and $y = \beta' x$ form a harmonic pencil, we must have [Art. 57]

$$\frac{(\alpha - \beta)(\alpha' - \beta')}{(\alpha - \beta')(\alpha' - \beta)} = -1,$$

or $$2\alpha\alpha' + 2\beta\beta' = (\alpha + \alpha')(\beta + \beta').$$

But, from the given equations we have

$$\alpha + \alpha' = -\frac{2h}{b}, \quad \alpha\alpha' = \frac{a}{b},$$

$$\beta + \beta' = -\frac{2h'}{b'}, \quad \beta\beta' = \frac{a'}{b'}.$$

Hence the condition required is

$$ab' + a'b = 2hh'.$$

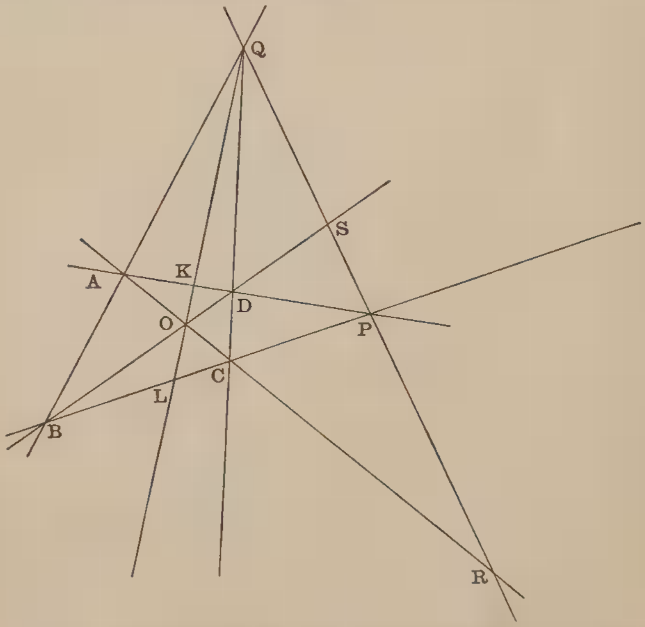
*59. We can shew in a similar manner that the pairs of points given by the equations

$$ax^2 + 2hx + b = 0, \text{ and } a'x^2 + 2h'x + b' = 0,$$

are harmonically conjugate if

$$ab' + a'b = 2hh'.$$

*60. *Each of the three diagonals of a quadrilateral is divided harmonically by the other two diagonals.*



Let the straight lines QAB, QDC, PDA and PCB be the sides of the quadrilateral. The line joining the point of intersection of two of these lines with the point of intersection of the other two is called a *diagonal* of the quadrilateral. There are therefore three diagonals, viz. PQ, AC, BD in the figure.

We have to prove that

$$\{AOCR\} = \{BODS\} = \{QSPR\} = -1.$$

Let QO cut AD in K and BC in L.

Then, from Art. 55,

$$\begin{aligned}\{AOCR\} &= Q\{AOCR\} = \{AKDP\}\\ &= O\{AKDP\} = \{CLBP\}\\ &= Q\{CLBP\} = \{COAR\}.\end{aligned}$$

And, since $\{AOCR\} = \{COAR\}$,

or $$\frac{AO \,.\, CR}{AR \,.\, CO} = \frac{CO \,.\, AR}{CR \,.\, AO};$$

$$\therefore \{AOCR\} = \pm 1.$$

We must take the *negative* sign, for two of the rays coincide if the anharmonic ratio of a pencil be equal to $+1$. This follows from Art. 55, for if $\dfrac{P''Q''}{R''Q''} = 1$, then P'' and R'' are coincident.

Hence the diagonal AC is cut harmonically.

We can prove in a similar manner that the other diagonals are divided harmonically.

INVOLUTION.

*61. *Def.* Let O be a fixed point on a given straight line, and P, P'; Q, Q'; R, R'; &c. pairs of points on the line such that

$$OP \,.\, OP' = OQ \,.\, OQ' = OR \,.\, OR' = \ldots\ldots = \text{a const.} = k.$$

Then these points are said to form a system in *involution*, of which the point O is called the *centre*. Two points such as P, P' are said to be *conjugate* to one another. The point conjugate to the centre is at an infinite distance.

If each point be on the same side of the centre as its conjugate, there will be two points K_1, K_2, one on each side of the centre, such that $OK_1^2 = OK_2^2 = OP \, . \, OP'$. These points K_1, K_2 are called *double points* or *foci*.

It is clear that when the two foci are given the involution is completely determined.

An involution is also completely determined when two pairs of conjugate points are given.

For, let a, a' and b, b' be the distances of these points from any point in the straight line upon which they lie, and let x be the distance of the centre of the involution from that point. Then we have the relation

$$(a-x)(a'-x) = (b-x)(b'-x),$$

or

$$(a+a'-b-b')\,x = aa' - bb'.$$

Hence there is only *one* position of the centre.

The position of the centre can be found geometrically by drawing circles one through each of the two pairs of conjugate points, then [Euclid III. 37] the common chord of the circles will cut the line on which the points lie in the required centre.

*62. *If any number of points be in involution the cross ratio of any four points is equal to that of their four conjugates.*

Let P, Q, R, S be any four points, and let the distances of these points from the centre be p, q, r, s respectively and therefore those of their conjugates $\frac{k}{p}, \frac{k}{q}, \frac{k}{r}, \frac{k}{s}$ respectively.

Then

$$\{PQRS\} = \frac{(q-p)(s-r)}{(s-p)(q-r)},$$

and

$$\{P'Q'R'S'\} = \frac{\left(\frac{k}{q}-\frac{k}{p}\right)\left(\frac{k}{s}-\frac{k}{r}\right)}{\left(\frac{k}{s}-\frac{k}{p}\right)\left(\frac{k}{q}-\frac{k}{r}\right)} = \frac{(p-q)(r-s)}{(p-s)(r-q)}.$$

Hence

$$\{PQRS\} = \{P'Q'R'S'\}.$$

The above gives us at once a means of testing whether or not *six* points are in involution. For P, P' will be conjugate points in the involution determined by A, A' and B, B', if $\{ABA'P\} = \{A'B'AP'\}$.

*63. *Any two conjugate points of an involution and the two foci form a harmonic range.*

Let K_1, K_2 be the two foci, and O the centre of the involution, and let $K_1O = c = OK_2$.

Then, if P, P' be the two conjugates we have to prove that

$$\frac{K_1P \,.\, K_2P'}{K_1P' \,.\, K_2P} = -1,$$

or $$(c + OP)(OP' - c) + (c + OP')(OP - c) = 0,$$

or $$OP \,.\, OP' = c^2,$$

which we know to be true.

*64. If any number of pairs of points in involution be joined to any point O we obtain a *pencil* of lines which may be said to be in involution.

Such a pencil is cut by any other transversal in pairs of points which are in involution. This follows from Articles 55 and 62.

EXAMPLES.

1. If P, Q, R, S be any four points on a straight line, then
$$PQ \,.\, RS + PR \,.\, SQ + PS \,.\, QR = 0.$$

2. Shew that
$$\{PQRS\} = \{QPSR\} = \{RSPQ\} = \{SRQP\}.$$

3. Shew that
$$\{PQRS\} = \frac{1}{\{PSRQ\}} = 1 - \{PRQS\}.$$

4. Shew that, by taking four points in different orders, six and only six different cross ratios are obtained, and that of these six three are the reciprocals of the other three.

5. If $\{PQRS\} = -1$, shew that $\{SRQP\} = -1$, $\{PRQS\} = 2$, and $\{PSQR\} = \frac{1}{2}$.

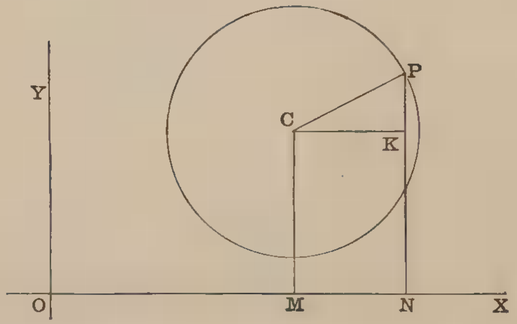
6. If $\{PQRS\} = -1$, and O be the middle point of PR, then
$$OP^2 = OQ \,.\, OS.$$

7. If $\{PQRS\} = -1$, shew that $\dfrac{1}{PQ} + \dfrac{1}{PS} = \dfrac{2}{PR}$, and $\dfrac{1}{RQ} + \dfrac{1}{RS} = \dfrac{2}{RP}$.

CHAPTER IV.

THE CIRCLE.

/ 65. *To find the equation of a circle referred to any rectangular axes.*



Let C be the centre of the circle, and P any point on its circumference. Let d, e be the co-ordinates of C; x, y the co-ordinates of P; and let a be the radius of the circle. Draw CM, PN parallel to OY, and CK parallel to OX, as in the figure. Then

$$CK^2 + KP^2 = CP^2.$$

But $CK = x - d$, and $KP = y - e$;

$$\therefore\ (x-d)^2 + (y-e)^2 = a^2 \dots\dots\dots\dots\dots\dots \text{(i)},$$

is the required equation.

If the centre of the circle be the origin, d and e will both be zero, and the equation of the circle will be

$$x^2 + y^2 = a^2 \dots\dots\dots\dots\dots\dots\dots\dots \text{(ii)}.$$

The equation (i) may be written

$$x^2 + y^2 - 2dx - 2ey + d^2 + e^2 - a^2 = 0.$$

The equation of any circle is therefore of the form

$$x^2 + y^2 + 2gx + 2fy + c = 0 \dots\dots\dots \text{(iii)},$$

where g, f and c are constants.

Conversely the equation (iii) is the equation of a circle.

For it may be written

$$(x+g)^2 + (y+f)^2 = g^2 + f^2 - c;$$

and this last equation shews that the distance from any point on the locus of the equation (iii) from the point $(-g, -f)$ is constant and equal to $\sqrt{g^2 + f^2 - c}$. The equation (iii) therefore represents a circle of radius $\sqrt{g^2 + f^2 - c}$, the centre of the circle being at the point $(-g, -f)$.

If $g^2 + f^2 - c = 0$ the radius of the circle is zero, and the circle is called a point-circle.

If $g^2 + f^2 - c$ be negative, no real values of x and of y will satisfy the equation, and the circle is called an imaginary circle.

66. We have seen that the general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

This equation contains *three* constants. If we want to find the equation of a circle which passes through three given points, or which is defined in some other manner, we assume the equation to be of the above form and determine the values of the constants g, f, c for the circle in question from the given conditions.

For example—To find the equation of the circle which passes through the three points (0, 1), (1, 0) and (2, 1).

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0.$$

Then, since (0, 1) is on the circle, the equation must be satisfied by putting $x=0$ and $y=1$;

$$\therefore 1+2f+c=0.$$

Also, since (1, 0) is on the curve,

$$1+2g+c=0.$$

And, since (2, 1) is on the curve,

$$4+1+4g+2f+c=0.$$

Whence $g=f=-1$, and $c=1$.

The required equation is therefore

$$x^2+y^2-2x-2y+1=0.$$

67. *To find the equation of a circle when the axes are inclined at an angle ω.*

The square of the distance of the point (x, y) from the point (d, e) will be equal to

$$(x-d)^2+(y-e)^2+2(x-d)(y-e)\cos\omega. \quad \text{[Art. 4.]}$$

Therefore the equation of the circle whose centre is at the point (d, e), and whose radius is a, will be

$$(x-d)^2+(y-e)^2+2(x-d)(y-e)\cos\omega=a^2 \ldots\ldots\text{(i)},$$

or $x^2+y^2+2xy\cos\omega-2x(d+e\cos\omega)-2y(e+d\cos\omega)$

$$+d^2+e^2+2de\cos\omega-a^2=0 \ldots\ldots\ldots\ldots\text{(ii)}.$$

Any circle therefore referred to oblique axes has its equation of the form

$$x^2+y^2+2xy\cos\omega+2gx+2fy+c=0 \ldots\ldots\text{(iii)},$$

where g, f, c are constants so long as we consider one particular circle, but are different for different circles.

The equation (iii) will still be true if we multiply throughout by any constant; it then takes the form

$$Ax^2+2A\cos\omega\, xy+Ay^2+2Gx+2Fy+C=0 \ldots\ldots\text{(iv)}.$$

Hence the equation of a circle referred to oblique axes is of the second degree, the coefficients of x^2 and y^2 are equal to one another, and the ratio of the coefficients of xy and x^2 is $2\cos\omega$, where ω is the angle between the axes.

We can find the centre and radius of the circle represented by the equation $x^2+y^2+2xy\cos\omega+2gx+2fy+c=0$. For it will be identical with $(x-d)^2+(y-e)^2+2(x-d)(y-e)\cos\omega-a^2=0$, if $d+e\cos\omega=-g$, $e+d\cos\omega=-f$, and $d^2+e^2+2de\cos\omega-a^2=c$. We therefore have $d\sin^2\omega=f\cos\omega-g$, $e\sin^2\omega=g\cos\omega-f$, and $a^2\sin^2\omega=f^2+g^2-2fg\cos\omega-c\sin^2\omega$.

EXAMPLES.

1. Find the radii and the co-ordinates of the centres of the circles whose equations are (i) $x^2+y^2-x-y=0$, (ii) $4x^2+4y^2+4x-8y+3=0$.

Ans. i. *centre* $(\frac{1}{2}, \frac{1}{2})$, *radius* $\frac{1}{\sqrt{2}}$; ii. *centre* $(-\frac{1}{2}, 1)$, *radius* $\frac{1}{\sqrt{2}}$.

2. Find the equation of the circle which passes through the points $(0, 0)$, $(a, 0)$ and $(0, b)$. *Ans.* $x^2+y^2-ax-by=0$.

3. Find the equation of the circle which passes through the points $(a, 0)$, $(-a, 0)$ and $(0, b)$.

Ans. $x^2+y^2+\dfrac{a^2-b^2}{b}y-a^2=0$.

4. Shew that, if the co-ordinates of the extremities of a diameter of a circle be (x', y') and (x'', y'') respectively, the equation of the circle will be $(x-x')(x-x'')+(y-y')(y-y'')=0$.

[The line joining any point P (x, y) on the circle to (x', y') makes with the axis of x an angle $\tan^{-1}\dfrac{y-y'}{x-x'}$, the line joining P to (x'', y'') makes an angle $\tan^{-1}\dfrac{y-y''}{x-x''}$. Since these lines are at right angles, we have

$$1+\frac{y-y'}{x-x'}\cdot\frac{y-y''}{x-x''}=0,$$

or

$$(x-x')(x-x'')+(y-y')(y-y'')=0].$$

5. Shew that if the co-ordinates of the extremities of a diameter be (x', y') and (x'', y'') respectively, the equation of the circle will be

$$(x-x')(x-x'')+(y-y')(y-y'')+\{(y-y')(x-x'')+(y-y'')(x-x')\}\cos\omega=0,$$

ω being the angle between the axes.

6. If the equation $x^2+xy+y^2+2x+2y=0$ represent a circle, shew that the axes are inclined at an angle of 60^0, and find the centre and radius of the circle.

Ans. centre $(-\frac{2}{3}, -\frac{2}{3})$; *radius* $\frac{2}{\sqrt{3}}$.

7. Find the equation of the circle through the three points (x', y'), (x'', y''), and (x''', y''').

68. *Def.* Let two points P, Q be taken on any curve, and let the point Q move along the curve nearer and nearer to the point P; then the limiting position of the line PQ, when Q moves up to and ultimately coincides with P, is called the *tangent* to the curve at the point P.

The line through the point P perpendicular to the tangent is called the *normal* to the curve at the point P.

69. *To find the equation of the tangent at any point of the circle whose equation is* $x^2+y^2=a^2$.

Let x', y' and x'', y'' be the co-ordinates of two points on the circle.

The equation of the secant through the points (x', y') and (x'', y'') is

$$\frac{x-x'}{x'-x''}=\frac{y-y'}{y'-y''}\text{..................(i).}$$

But, since the two points are on the circle, we have

$$x'^2+y'^2=a^2, \text{ and } x''^2+y''^2=a^2;$$

$$\therefore\ x'^2-x''^2=y''^2-y'^2\text{..............(ii).}$$

Multiply the corresponding sides of the equations (i) and (ii), and we have

$$(x-x')(x'+x'')=-(y-y')(y'+y'')\text{......(iii).}$$

Let (x'', y'') move up to and ultimately coincide with (x', y'); then in the limit the chord becomes the tangent at (x', y'). The equation of the tangent at (x', y') is therefore obtained by putting $x''=x'$, and $y''=y'$ in equation (iii); the result is

$$(x-x')x'+(y-y')y'=0,$$

or
$$xx' + yy' = x'^2 + y'^2;$$
$$\therefore\ xx' + yy' = a^2$$
is the required equation of the tangent at the point (x', y').

70. *To find the equation of the tangent at any point of the circle whose equation is*
$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

The equation of the secant through the two points (x', y'), (x'', y'') will be
$$\frac{x - x'}{x' - x''} = \frac{y - y'}{y' - y''} \text{.....................(i).}$$

Since the two points are on the circle, we have
$$x'^2 + y'^2 + 2gx' + 2fy' + c = 0,$$
$$x''^2 + y''^2 + 2gx'' + 2fy'' + c = 0,$$
$$\therefore\ (x' - x'')(x' + x'' + 2g) = -(y' - y'')(y' + y'' + 2f) \text{...(ii).}$$

Multiply the corresponding sides of the equations (i) and (ii), and we get for the equation of the secant
$$(x - x')(x' + x'' + 2g) = -(y - y')(y' + y'' + 2f).$$

The equation of the tangent at (x', y') will therefore be
$$(x - x')(x' + g) + (y - y')(y' + f) = 0,$$
or
$$xx' + yy' + gx + fy = x'^2 + y'^2 + gx' + fy'.$$

Add $gx' + fy' + c$ to both sides; then, since (x', y') is on the circle, the equation of the tangent becomes
$$xx' + yy' + g(x + x') + f(y + y') + c = 0.$$

71. *To find the equation of the normal at any point of a circle.*

Let the equation of the circle be
$$x^2 + y^2 = a^2.$$

If (x', y') be any point on the circle, the equation of the tangent at that point will be
$$xx' + yy' = a^2 \text{.....................(i).}$$

The equation of the line through (x', y') perpendicular to (i) is [Art. 30]
$$(x - x')y' - (y - y')x' = 0,$$

or $$xy' - yx' = 0 \text{.....................(ii).}$$
This is the required equation of the normal at (x', y').

It is clear from equation (ii) that the normal at any point of the circle passes through the origin, that is through the centre of the circle.

72. *To find the points of intersection of a given straight line and a circle.*

Let the equation of the circle be
$$x^2 + y^2 = a^2 \text{ ....(i),}$$
and let the equation of the straight line be
$$y = mx + c \text{.......................(ii).}$$
At points which are common to the straight line and the circle both these relations are satisfied. Points on the straight line satisfy the equation $y^2 = (mx + c)^2$, and points on the circle satisfy the equation $y^2 = a^2 - x^2$; hence for the common points we have
$$(mx + c)^2 = a^2 - x^2,$$
or $$x^2(1 + m^2) + 2mcx + c^2 - a^2 = 0 \text{.........(iii).}$$
This is a quadratic equation, and every quadratic equation has *two* roots, real, coincident or imaginary.

Hence there are *two* values of x, and the two corresponding values of y are found from (ii). So that every straight line meets a circle in two real, coincident, or imaginary points—imaginary points being those one or both of whose co-ordinates are imaginary.

It is impossible to represent geometrically the two imaginary points of intersection of a straight line and a circle: we shall find however that imaginary points and lines have often an important significance: and it is necessary to consider them in order to enunciate our theorems in their most general forms.

The roots of the equation (iii) will be *equal* to one another, if
$$(1 + m^2)(c^2 - a^2) = m^2c^2,$$
that is, if $$c^2 = a^2(1 + m^2) \text{ ... .(iv).}$$

If the two values of x are equal to one another the two values of y must also be equal to one another from (ii).

Therefore the two points in which the circle is cut by the line will be *coincident* if $c = a\sqrt{1+m^2}$.

Hence the line $y = mx + a\sqrt{1+m^2}$ will *touch* the circle $x^2 + y^2 = a^2$ for all values of m.

Since either sign may be given to the radical $\sqrt{1+m^2}$, it follows that there are *two* tangents to a circle for every value of m, that is, there are two tangents parallel to any given straight line.

73. *To find the locus of the middle points of a system of parallel chords of a circle.*

Take the centre of the circle for origin, and the axis of x parallel to the chords.

Let the equation of the circle be

$$x^2 + y^2 = a^2 \text{.....................(i)};$$

and let the equation of any one of the parallel chords be

$$y - c = 0 \text{ ....(ii)}.$$

Where (i) and (ii) meet we have

$$x^2 + c^2 = a^2;$$

$$\therefore\ x = \pm\sqrt{a^2 - c^2}.$$

Since the two values of x are equal and opposite, it follows that the middle point of the chord has its abscissa zero, that is, the middle point of the chord is always on the axis of y. This is true for *all* values of c. If $c > a$ the two values of x are both *imaginary*, but their sum is still zero, and therefore the middle point of the chord is still on the axis of y.

The locus of the middle points of parallel chords of a circle is therefore the straight line through the centre which is perpendicular to the chords: the locus need not however be supposed to be limited to that portion of this line which is *within* the circle.

74. In the preceding Articles we have assumed no geometrical properties of the circle except that the distance from any point to the centre is constant. Some of our results may be obtained more readily by assuming the propositions proved in Euclid, Book III. For instance, let (x', y') be any point on the circle whose equation is $x^2+y^2=a^2$; the equation of the line from (x', y') to the centre of the circle is $\frac{x}{x'}-\frac{y}{y'}=0$, and the equation of a perpendicular line through (x', y') is [Art. 30]

$$(x-x')\,x'+(y-y')\,y'=0 \text{ or } xx'+yy'-a^2=0.$$

And by Euclid III. this line is the tangent at the point.

75. *Two tangents can be drawn to a circle from any point; and these two tangents will be real if the point be outside the circle, coincident if the point be on the circle, and imaginary if the point be within the circle.*

Let the equation of the circle be

$$x^2+y^2=a^2,$$

and let h, k be the co-ordinates of any point. Let x', y' be the co-ordinates of any point on the circle, then the equation of the tangent at (x', y') will be

$$xx'+yy'=a^2.$$

The tangent at (x', y') will pass through the point (h, k) if

$$hx'+ky'=a^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

But (x', y') is on the circle, therefore

$$x'^2+y'^2=a^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Equations (i) and (ii) determine the values of x' and of y' for the points the tangents at which pass through the particular point (h, k). Substitute for y' in (ii) and we get

$$x'^2+\left(\frac{a^2-hx'}{k}\right)^2=a^2,$$

or $$x'^2(h^2+k^2)-2a^2hx'+a^2(a^2-k^2)=0 \;\ldots\ldots\text{(iii)}.$$

Equation (iii) gives the abscissae, and from (i) we get the corresponding ordinates. Since equation (iii) is a *quadratic* equation, there are *two* points the tangents at which pass through (h, k).

The roots of (iii) are real, coincident, or imaginary according as

$$a^4h^2 - a^2(a^2 - k^2)(h^2 + k^2)$$

is greater than, equal to, or less than zero.

That is, according as

$$h^2 + k^2 - a^2$$

is greater than, equal to, or less than zero. That is, according as (h, k) is outside the circle, on the circle, or within the circle.

EXAMPLES.

1. Find the co-ordinates of the points where the line $y = 2x + 1$ cuts the circle $x^2 + y^2 = 2$. *Ans.* $(-1, -1)$ *and* $(\frac{1}{5}, \frac{7}{5})$.

2. Shew that the line $3x - 2y = 0$ touches the circle $x^2 + y^2 - 3x + 2y = 0$.

3. Shew that the circles $x^2 + y^2 = 2$ and $x^2 + y^2 - 6x - 6y + 10 = 0$ touch one another at the point (1, 1).

4. Shew that the circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ touches the axes of x and y.

5. Find the equation of the circle which touches the lines $x = 0$, $y = 0$, and $x = c$. *Ans.* $4x^2 + 4y^2 - 4cx \pm 4cy + c^2 = 0$.

6. Find the equation of the circle which touches the lines $x = 0$, $x = a$, and $3x + 4y + 5a = 0$.

Ans. $x^2 + y^2 - ax + 2ay + a^2 = 0$ or $x^2 + y^2 - ax + \frac{9}{2}ay + \frac{81}{16}a^2 = 0$.

7. Shew that the line $y = m(x - a) + a\sqrt{1 + m^2}$ touches the circle $x^2 + y^2 = 2ax$, whatever the value of m may be.

8. Two lines are drawn through the points $(a, 0)$, $(-a, 0)$ respectively, and make an angle θ with one another; find the locus of their intersection.

The circles $x^2 + y^2 - a^2 = \pm 2ay \cot\theta$.

9. A circle touches one given straight line and cuts off a constant length $(2l)$ from another straight line perpendicular to the former; find the equation of the locus of its centre. *Ans.* $y^2 - x^2 = l^2$.

10. A line moves so that the sum of the perpendiculars drawn to it from the points $(a, 0)$, $(-a, 0)$ is constant; shew that it always touches a circle.

11. Find the equations of the two tangents to $x^2+y^2=3$, which make an angle of 60^0 with the axis of x. *Ans.* $y=\sqrt{3}\,(x\pm 2)$.

12. Find the equation of the circle inscribed in the triangle the equations of whose sides are $x=1$, $2y=5$ and $3x-4y=5$.

Ans. $(x-2)^2+(y-\frac{3}{2})^2=1$.

13. Shew that the two circles

$$x^2+y^2-2ax-2by-2ab=0 \text{ and } x^2+y^2+2bx+2ay-2ab=0$$

cut one another at right angles. [This requires that the square of the distance between the centres of the circles is equal to the sum of the squares of their radii.]

14. Shew that the two circles represented by the equations

$$x^2+y^2+2dx+k^2=0,\quad x^2+y^2+2d'y-k^2=0$$

intersect at right angles.

76. *Tangents are drawn to a circle from any point; to find the equation of the straight line joining the points of contact of the tangents.*

Let the co-ordinates of the point from which the tangents are drawn be x' y'. Let the co-ordinates of the two points of contact be h, k and h', k', and let $x^2+y^2-a^2=0$ be the equation of the circle.

The equations of the two tangents will be [Art. 69]

$$xh+yk-a^2=0,$$
$$xh'+yk'-a^2=0.$$

Since both these tangents pass through the point (x', y'), therefore both equations are satisfied by the co-ordinates x', y';

$$\therefore\ x'h+y'k-a^2=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

and

$$x'h'+y'k'-a^2=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

But the equations (i) and (ii) are the conditions that the two points (h, k) and (h', k') may lie on the line whose equation is

$$x'x+y'y-a^2=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Hence (iii) is the required equation of the straight line through the two points of contact of the tangents which pass through $(x'\ y')$.

If the equation of the circle be $x^2+y^2+2gx+2fy+c=0$, we can shew in a similar manner (by assuming the result of Art. 70) that the equation of the line joining the points of contact of the tangents which pass through (x', y') is

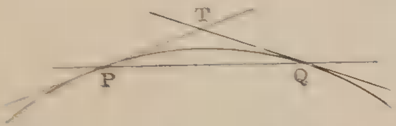
$$xx' + yy' + g(x+x') + f(y+y') + c = 0.$$

If the point (x', y') be *outside* the circle the two tangents will be real, and the co-ordinates h, k and h', k' will all be real. If however the point (x', y') be *within* the circle the two tangents will be imaginary; but, even in this case, the line whose equation is (iii) is a real line when x' and y' are real. So that there is a *real* line joining the *imaginary* points of contact of the two *imaginary tangents* which can be drawn from a point within the circle.

Def. The straight line through the points of contact of the tangents (real or imaginary) which can be drawn from any point to a circle is called the *polar* of that point with respect to the circle.

The point of intersection of the tangents to a circle at the (real or imaginary) points of intersection of the circle and a straight line is called the *pole* of that line with respect to the circle.

77. Let TP, TQ be the two tangents to a circle from any point T. Let Q move up to and ultimately coincide with the point P, then T will also move up to and ultimately coincide with P, and the tangents TP, TQ will ultimately coincide with one another and with the chord PQ. That is to say, the polar of T, when T is on the circle, coincides with the tangent at that point.



This agrees with the result of Art. 76. For the equation of the polar is of the same form as the equation of the tangent, and hence the polar of a point which is on the circle is the tangent at that point.

78. *If the polar of a point P pass through Q, then will the polar of Q pass through P.*

Let P be the point (x', y'), and Q be the point (x'', y''), and let the equation of the circle be $x^2 + y^2 - a^2 = 0$.

The equations of the polars of (x', y') and (x'', y'') are

$$xx' + yy' - a^2 = 0 \qquad \text{(i)},$$

and

$$xx'' + yy'' - a^2 = 0 \qquad \text{(ii)}.$$

If Q be on the polar of P, its co-ordinates must satisfy the equation (i);

$$\therefore\ x''x' + y''y' - a^2 = 0;$$

but this is also the condition that P may be on the line (ii), that is on the polar of Q, which proves the proposition.

If Q be any point on a fixed straight line, and P be the pole of that line; then the polar of Q must pass through P, for by supposition the polar of P passes through Q.

Conversely, if through a fixed point P any straight line be drawn, and Q be the pole of that line; then, since P is on the polar of Q, the point Q must always lie on a fixed straight line, namely on the polar of P.

79. If the polars of two points P, Q meet in R, then R is the pole of the line PQ. For R is on the polar of P, therefore, by Art. 78, the polar of R goes through P; similarly it goes through Q; and therefore it must be the line PQ.

80. *To give a geometrical construction for the polar of a point with respect to a circle.*

Let the equation of the circle be

$$x^2 + y^2 = a^2;$$

let P be any point, and let the co-ordinates of P be x', y'.

The equation of the polar of P with respect to the circle is

$$xx' + yy' - a^2 = 0 \qquad \text{(i)}.$$

The equation of the line joining P to O, the centre of the circle, is

$$\frac{x}{x'} - \frac{y}{y'} = 0 \qquad \text{(ii)}.$$

We see from the equations (i) and (ii) that the polar of any point with respect to a circle is perpendicular to the line joining the point to the centre of the circle.

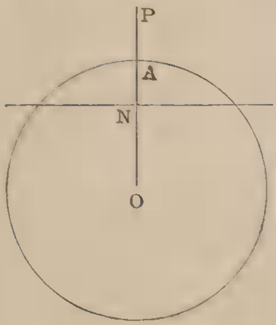
If ON be the perpendicular from O on the polar,

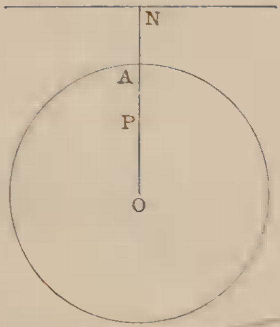
$$ON = \frac{a^2}{\sqrt{x'^2 + y'^2}}; \qquad \text{[Art. 31.]}$$

also $$OP = \sqrt{x'^2 + y'^2};$$

therefore $$ON \,.\, OP = a^2.$$

We have therefore the following construction for the polar. Join OP and let it cut the circle in A; take N on the line OP such that $OP : OA :: OA : ON$, and draw through N a line perpendicular to OP.





Ex. 1. Write down the polars of the following points with respect to the circle whose equation is $x^2 + y^2 = 4$,

(i) $(2, 3)$, (ii) $(3, -1)$, (iii) $(1, -1)$.

Ex. 2. Find the poles of the following lines with respect to the circle whose equation is $x^2 + y^2 = 35$,

(i) $4x + 6y - 7 = 0$, (ii) $3x - 2y - 5 = 0$, (iii) $ax + by - 1 = 0$.

Ans. (i) $(20, 30)$, (ii) $(21, -14)$, (iii) $(35a, 35b)$.

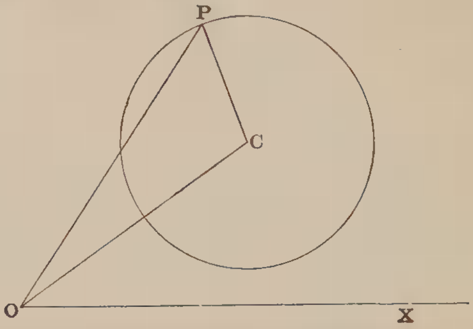
Ex. 3. Find the co-ordinates of the points where the line $x = 4$ cuts the circle $x^2 + y^2 = 4$; find the equations of the tangents at those points, and shew that they intersect in the point $(1, 0)$.

Ans. $(4, \pm\sqrt{-12})$, $4x \pm \sqrt{-12}\,y = 4$.

Ex. 4. If the polar of the point x', y' with respect to the circle $x^2 + y^2 = a^2$ touch the circle $(x - a)^2 + y^2 = a^2$, shew that $y'^2 + 2ax' = a^2$.

81. *To find the polar equation of a circle.*

Let C be the centre of the circle, and let its polar co-ordinates be ρ, α, and let the radius of the circle be equal to a.



Let the polar co-ordinates of any point P on the curve be r, θ.

Then $$CP^2 = OC^2 + OP^2 - 2OC \,.\, OP \cos COP.$$

But $CP = a$, $OC = \rho$, $OP = r$, $\angle XOC = \alpha$, $\angle XOP = \theta$;

$$\therefore\ a^2 = \rho^2 + r^2 - 2r\rho \cos(\theta - \alpha) \ldots\ldots\ldots \text{(i)},$$

which is the required equation.

If the origin be on the circumference of the circle $\rho = a$, and we have from (i)

$$r = 2a \cos(\theta - \alpha) \ \ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

If, in addition, the initial line pass through the centre, α will be zero, and the equation will be

$$r = 2a \cos\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}.$$

From equation (i) we see that if r_1, r_2 be the two values of r corresponding to any particular value of θ, then

$$r_1\, r_2 = \rho^2 - a^2 \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)},$$

so that $r_1\, r_2$ is independent of θ.

This proves that, if from a fixed point a straight line be drawn to cut a given circle, the rectangle contained by the segments is constant.

From (iv) we see that if the origin be within the circle, in which case ρ is less than a, r_1 and r_2 must have different signs, and are therefore drawn in different directions, as is geometrically obvious.

82. *To find the length of the tangent drawn from a given point to a circle.*

If T be the given point, and TP be one of the tangents from T to the circle whose centre is C, then we know that the angle CPT is a right angle;

$$\therefore\ TP^2 = CT^2 - CP^2 \ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Let the equation of the circle be

$$(x-a)^2 + (y-b)^2 - c^2 = 0 \ldots\ldots\ldots\ldots\text{(ii)},$$

and let the co-ordinates of T be x', y'.

Then $$CT^2 = (x'-a)^2 + (y'-b)^2;$$

therefore from (i) we have

$$TP^2 = (x'-a)^2 + (y'-b)^2 - c^2 \ldots\ldots\ldots\text{(iii)}.$$

TP^2 is therefore found by substituting the co-ordinates x', y' in the left-hand member of the equation (ii).

We see, therefore, that if $S = 0$ be the equation of a circle (where S is written for shortness instead of $x^2 + y^2 + 2gx + 2fy + c$), and the co-ordinates of any point be substituted in S, the result is equal to the square of the length of the tangent drawn from that point to the circle; or [Euclid III. 37] to the rectangle of the segments of chords drawn through the point. If the point be *within* the circle the rectangle is negative, and the length of the tangent imaginary.

If the equation of the circle be

$$Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0,$$

to find the square of the length of the tangent from any point to the circle we must divide by A and then substitute the co-ordinates of the point from which the tangent is drawn.

83. If $$x^2 + y^2 + 2gx + 2fy + c = 0 \ldots\ldots\text{(i)}$$

be the equation of one circle,

and $$x^2 + y^2 + 2g'x + 2f'y + c' = 0 \ldots\ldots\text{(ii)}$$

be the equation of another circle, the equation

$$x^2 + y^2 + 2gx + 2fy + c = x^2 + y^2 + 2g'x + 2f'y + c' \ldots\text{(iii)}$$

will clearly be satisfied by the co-ordinates of any point which is on (i) and also on (ii). Equation (iii) represents therefore some locus passing through the points common to the two circles.

But (iii) reduces to

$$2(g-g')x+2(f-f')y+c-c'=0 \ldots\ldots (\text{iv}),$$

which is of the first degree, and therefore represents a *straight line.*

Hence (iii), or (iv), is the equation of the straight line through the points common to the circles (i) and (ii).

Although the two circles (i) and (ii) may not cut one another in real points, the straight line given by (iii) or by (iv) is in all cases real, provided that g, f, c, g', f', c' are real. We have here therefore the case of a real straight line which passes through the imaginary points of intersection of two circles.

Another geometrical meaning can however be given to the equation (iii).

For if $S=0$ be the equation of a circle, in which the coefficient of x^2 is unity, and the co-ordinates of any point be substituted in S, the result is equal to the square of the tangent drawn from that point to the circle $S=0$. [Art. 82.]

Now if x, y be the co-ordinates of any point on the line (iii) the left side of that equation is equal to the square of the tangent from (x, y) to the circle (i), and the right side is equal to the square of the tangent from (x, y) to the circle (ii).

Hence the tangents drawn to the two circles from any point of the line (iii) are equal to one another.

Def. The straight line through the (real or imaginary) points of intersection of two circles is called the *radical axis* of the two circles.

From the above we see that the *radical axis* of two circles may also be defined as the locus of the points from which the tangents drawn to the two circles are equal to one another.

The co-ordinates of the centres of the two circles are $-g, -f$ and $-g', -f'$ respectively: the equation of the line joining them, therefore is

$$\frac{x+g}{g-g'}=\frac{y+f}{f-f'},$$

which [Art. 30] is perpendicular to the line (iv).

Hence the radical axis of two circles is perpendicular to the line joining their centres. This is geometrically obvious when the circles cut in real points.

84. *The three radical axes of three circles taken in pairs meet in a point.*

If $S=0$, $S'=0$, $S''=0$ be the equations of three circles (in each of which the coefficient of x^2 is unity), the equation of the radical axis of the first and second will be

$$S-S'=0.$$

The equation of the radical axis of the second and third will be

$$S'-S''=0.$$

And of the third and first will be

$$S''-S=0.$$

And it is obvious that if two of these equations be satisfied by the co-ordinates of any point, the third equation will also be satisfied by those co-ordinates.

The point of intersection of the three radical axes is called the *radical centre* of the three circles.

*85. *To find the equation of a system of circles every pair of which has the same radical axis.*

If the common radical axis be taken for the axis of y, and a line perpendicular to it for the axis of x, then all the circles cut $x=0$ in the same two points.

Hence, if $x^2+y^2+2gx+2fy+c=0$ be the general equation of the circles, when we put $x=0$ the roots of the resulting equation $y^2+2fy+c=0$ must be the same for all the circles.

Therefore f and c must be the same for all the circles.

If we take as origin the point midway between the two points where $x=0$ cuts the circles, f will be zero, and the equation becomes

$$x^2+y^2+2gx+c=0\text{..................(i)},$$

which is the required equation, c being the same for all the circles.

The radical axis cuts the circles in *real* points if c be negative, and in *imaginary* points if c be positive.

The equation (i) can be written

$$(x+g)^2+y^2=g^2-c.$$

Hence, if g be taken equal to $\pm\sqrt{c}$ the circle will reduce to one of the *points* $(\mp\sqrt{c},\ 0)$.

These points are called the *limiting points* of the system of co-axial circles. When c is positive, that is when the circles themselves cut in imaginary points, the limiting points are real, and conversely, when the circles cut in real points the limiting points are imaginary.

*86. *If* $S=0$ *and* $S'=0$ *be the equations of two circles,* $S-\lambda S'=0$ *will, for different values of* λ, *represent all circles which pass through the points common to* $S=0$ *and* $S'=0$.

For, if $S=0$ and $S'=0$ be

$$x^2+y^2+2gx+2fy+c=0\text{..........(i)},$$
$$x^2+y^2+2g'x+2f'y+c'=0\text{.........(ii)},$$

then will $S-\lambda S'=0$ be

$$x^2+y^2+2gx+2fy+c-\lambda\{x^2+y^2+2g'x+2f'y+c'\}=0\text{.........(iii)}.$$

Now (iii) is clearly the equation of a circle, whatever λ may be.

Also, if the co-ordinates of any point satisfy both (i) and (ii), they will also satisfy (iii).

Hence $S-\lambda S'=0$ is, for any value of λ, a circle passing through the points common to $S=0$ and $S'=0$.

By giving a suitable value to λ the circle (iii) may be made to pass through any other point; therefore $S-\lambda S'=0$ represents all the circles through the intersections of $S=0$ and $S'=0$.

The geometrical meaning of the equation $S - \lambda S' = 0$ should be noticed. From Art. 82 we see that any point whose co-ordinates satisfy the equation $S = \lambda S'$ is such that the square of the tangent from it to the circle $S = 0$ is equal to λ times the square of the tangent from it to $S' = 0$. We have therefore the following proposition—the locus of a point which moves so that the tangents from it to two given circles are in a constant ratio, is another circle which passes through their common points.

87. If O, O' be the centres of two circles whose radii are a, a' respectively, the two points which divide the line OO' internally and externally in the ratio $a : a'$ are called the *centres of similitude* of the two circles.

The properties of the centres of similitude are best treated geometrically.

The most important of the properties are (1) Two of the common tangents to two circles pass through each centre of similitude; (2) Any straight line through a centre of similitude of two circles is cut similarly by the two circles.

EXAMPLES.

1. Find the length of the tangent drawn from the point (2, 5) to the circle $x^2 + y^2 - 2x - 3y - 1 = 0$.

Also the length of the tangents from (4, 1) to the circle

$$4x^2 + 4y^2 - 3x - y - 7 = 0. \qquad \textit{Ans. } 3,\ 2\sqrt{3}.$$

2. Find the equation of the circle through the points (3, 0), (0, 2) and (– 1, 1); and find the value of the constant rectangle of the segments of all chords through the origin. *Ans.* $\frac{18}{6}$.

3. Find the radical axis of the circles $x^2 + y^2 + 2x + 3y - 7 = 0$ and $x^2 + y^2 - 2x - y + 1 = 0$. *Ans.* $x + y - 2 = 0$.

4. Find the radical axis of the two circles $x^2 + y^2 + bx + by - c = 0$ and $ax^2 + ay^2 + a^2x + b^2y = 0$.

$$\textit{Ans. } ax - by + \frac{ca}{a - b} = 0.$$

5. Find the radical axis and the length of the common chord of the circles $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + bx + ay + c = 0$.

Ans. $x - y = 0$, $\{\frac{1}{2}(a + b)^2 - 4c\}^{\frac{1}{2}}$.

6. Shew that the three circles $x^2+y^2+3x+6y+12=0$, $x^2+y^2+2x+8y+16=0$, and $x^2+y^2+12y+24=0$, have a common radical axis.

7. Find the radical centre of the three circles

$$x^2+y^2+4x+7=0,\ \ 2x^2+2y^2+3x+5y+9=0, \text{ and } x^2+y^2+y=0.$$

Ans. $(-2, -1)$.

8. Find the equations of the straight lines which touch both the circles $x^2+y^2=4$ and $(x-4)^2+y^2=1$. Find also the co-ordinates of the centres of similitude.

Ans. $3x \pm \sqrt{7}y-8=0$, and $x \pm \sqrt{15}y-8=0$; $(8, 0)$, $(\frac{8}{3}, 0)$.

9. If the length of the tangent from (f, g) to the circle $x^2+y^2=6$ be twice the length of the tangent from (f, g) to the circle $x^2+y^2+3x+3y=0$, then will $f^2+g^2+4f+4g+2=0$.

10. If the length of the tangent from any point to the circle $x^2+y^2+2x=0$ be three times the length of the tangent from the same point to the circle $x^2+y^2-4=0$, shew that the point must be on the circle $4x^2+4y^2-x-18=0$.

11. Find the equation of the circle through the points of intersection of the circles $x^2+y^2+2x+3y-7=0$ and $x^2+y^2+3x-2y-1=0$, and through the point $(1, 2)$. *Ans.* $x^2+y^2+4x-7y+5=0$.

12. Find the equation of a circle through the points of intersection of $x^2+y^2-4=0$ and $x^2+y^2-2x-4y+4=0$ and touching the line $x+2y=0$.

Ans. $x^2+y^2-x-2y=0$.

*88. We shall conclude this chapter by the solution of some examples.

(1) *To find the equation of the circle which cuts three given circles at right angles.*

Let the equations of the given circles be

$$x^2+y^2+2gx+2fy+c=0 \text{..................... (i)},$$

$$x^2+y^2+2g'x+2f'y+c'=0 \text{..................... (ii)},$$

$$x^2+y^2+2g''x+2f''y+c''=0 \text{..................... (iii)}.$$

If the circle whose equation is

$$x^2+y^2+2Gx+2Fy+C=0 \text{..................... (iv)}$$

cut (i) at right angles, the square of the distance between their centres is equal to the sum of the squares of their radii. Hence we have

$$(G-g)^2+(F-f)^2=G^2+F^2-C+g^2+f^2-c,$$

or

$$2Gg+2Ff-C-c=0 \text{..................... (v)}.$$

We also have, since the other circles are cut at right angles,

$$2Gg'+2Ff'-C-c'=0 \text{..................... (vi)},$$

$$2Gg''+2Ff''-C-c''=0 \text{..................... (vii)}.$$

Eliminating G, F, C, from the equations (iv), (v), (vi), and (vii), we have for the required equation

$$\begin{vmatrix} x^2+y^2, & x, & y, & 1 \\ -c, & g, & f, & -1 \\ -c', & g', & f', & -1 \\ -c'', & g'', & f'', & -1 \end{vmatrix} = 0.$$

(2) *The polars of any fixed point with respect to a series of co-axial circles pass through another fixed point, and the polar of one of the limiting points of the system is the same for all the circles.*

The system of circles is given by the equation

$$x^2+y^2+2ax+c=0 \text{..........................(i)},$$

where c is the same for all the circles [Art. 85].

The limiting points of the system are $(\pm\sqrt{c}, 0)$.

Let the co-ordinates of the fixed point be (f, g), then the equation of the polar with respect to (i) will be

$$xf+yg+a(x+f)+c=0 \text{........................(ii)}.$$

And, whatever the value of a may be, the straight line (ii) always passes through the point given by $xf+yg+c=0$ and $x+f=0$.

If $f=\pm\sqrt{c}$ and $g=0$, equation (ii) reduces to $f(x+f)+a(x+f)=0$; and therefore $x+f=0$.

Hence the polar of one of the limiting points is the line through the other limiting point parallel to the radical axis.

(3) *If* ABC *be any triangle, and* A′B′C′ *be the triangle formed by the polars of the three points* A, B, C *with respect to a circle, so that* B′C′ *is the polar of* A, C′A′ *is the polar of* B, *and* A′B′ *is the polar of* C; *then will the three lines* AA′, BB′, CC′ *meet in a point.*

Let the equation of the circle be

$$x^2+y^2=a^2 \text{..............................(i)},$$

and let the co-ordinates of the points A, B, C be x', y'; x'', y''; and x''', y''' respectively.

Then the equations of the three lines $B'C'$, $C'A'$, $A'B'$ will be

$$xx'+yy'-a^2=0 \text{............................(ii)},$$

$$xx''+yy''-a^2=0 \text{............................(iii)},$$

and

$$xx'''+yy'''-a^2=0 \text{..........................(iv)}.$$

AA' is a line through the intersection of (iii) and (iv), its equation is therefore [Art. 33] included in

$$xx''+yy''-a^2=\lambda(xx'''+yy'''-a^2).$$

We find λ by making the above line pass through A, whose co-ordinates are x', y'; we get therefore

$$x'x''+y'y''-a^2=\lambda(x'x'''+y'y'''-a^2).$$

Hence the equation of AA' is

$$(xx''+yy''-a^2)(x'''x'+y'''y'-a^2)-(xx'''+yy'''-a^2)(x'x''+y'y''-a^2)=0\ldots(\text{v}).$$

The other equations can now be written down from symmetry.

They will be

$$(xx'''+yy'''-a^2)(x'x''+y'y''-a^2)-(xx'+yy'-a^2)(x''x'''+y''y'''-a^2)=0\ldots(\text{vi}),$$

and

$$(xx'+yy'-a^2)(x''x'''+y''y'''-a^2)-(xx''+yy''-a^2)(x'''x'+y'''y'-a^2)=0\ldots(\text{vii}).$$

Since the three equations (v), (vi), (vii) when added together vanish identically, the three lines AA', BB', CC' represented by those equations must meet in a point. [Art. 34.]

(4) *O is one of the points of intersection of two given circles, and any line through* O *cuts the circles again in the points* P, Q *respectively. Find the locus of the middle point of* PQ.

Let O be taken for origin, and let the equations of the circles be [Art. 81]

$$r=2a\cos(\theta-\alpha),\text{ and } r=2b\cos(\theta-\beta).$$

Then, for any particular value of θ,

$$OP=2a\cos(\theta-\alpha),$$

and

$$OQ=2b\cos(\theta-\beta).$$

If R be the middle point of PQ,

$$OR=\tfrac{1}{2}(OP+OQ);$$

$$\therefore\quad OR=a\cos(\theta-\alpha)+b\cos(\theta-\beta).$$

The locus of R is therefore given by

$$r=a\cos(\theta-\alpha)+b\cos(\theta-\beta)$$
$$=(a\cos\alpha+b\cos\beta)\cos\theta+(a\sin\alpha+b\sin\beta)\sin\theta.$$

The locus is therefore the circle whose equation is

$$r=A\cos(\theta-B),$$

where A and B are given by the equations

$$A\cos B=a\cos\alpha+b\cos\beta,\text{ and } A\sin B=a\sin\alpha+b\sin\beta.$$

(5) *If from any point* O *on the circle circumscribing a triangle* ABC, *perpendiculars be drawn on the sides of the triangle, the feet of these perpendiculars will lie on a straight line.*

Take the point O for origin, and the diameter through it for initial line, then the equation of the circle will be $r=2a\cos\theta$.

Let the angular co-ordinates of the points A, B, C be α, β, γ respectively.

The line BC is the line joining $(2a\cos\beta, \beta)$ and $(2a\cos\gamma, \gamma)$. To find the polar equation of BC take the general form $p=r\cos(\theta-\phi)$ [Art. 45] and substitute the co-ordinates of B and of C. We thus obtain two equations to determine p and ϕ. The equations will be

$$p=2a\cos\beta\cos(\beta-\phi),\text{ and } p=2a\cos\gamma\cos(\gamma-\phi).$$

Hence $\phi=\beta+\gamma$, and $p=2a\cos\beta\cos\gamma$. The equation of BC is therefore

$$2a\cos\beta\cos\gamma=r\cos(\theta-\beta-\gamma)\text{.........}\quad\text{..........(i).}$$

Similarly, the equations of CA and of AB will be respectively

$$2a\cos\gamma\cos\alpha=r\cos(\theta-\gamma-\alpha)\text{.....................(ii),}$$

and

$$2a\cos\alpha\cos\beta=r\cos(\theta-\alpha-\beta)\text{....................(iii).}$$

The co-ordinates of the feet of the perpendiculars on the lines (i), (ii), (iii), from the point O, are $2a\cos\beta\cos\gamma$, $\beta+\gamma$; $2a\cos\gamma\cos\alpha$, $\gamma+\alpha$; and $2a\cos\alpha\cos\beta$, $\alpha+\beta$. These three points are all on the straight line whose equation is

$$2a\cos\alpha\cos\beta\cos\gamma=r\cos(\theta-\alpha-\beta-\gamma)\text{..............(iv).}$$

The line through the feet of the perpendiculars is called the *pedal line* of the point O with respect to the triangle.

Let D be another point on the circle, and let the angular co-ordinate of D be δ. The four points A, B, C, D can be taken in threes in four ways, and we shall have four pedal lines of O corresponding to the four triangles. We have found the equation of one of these pedal lines, viz. equation (iv). The equations of the others can be written down by symmetry; they will be

$$2a\cos\beta\cos\gamma\cos\delta=r\cos(\theta-\beta-\gamma-\delta)\text{..............(v),}$$

$$2a\cos\gamma\cos\delta\cos\alpha=r\cos(\theta-\gamma-\delta-\alpha)\text{.............(vi),}$$

and

$$2a\cos\delta\cos\alpha\cos\beta=r\cos(\theta-\delta-\alpha-\beta)\text{............(vii).}$$

The co-ordinates of the feet of the perpendiculars from O on the lines (iv), (v), (vi) and (vii) will be $2a\cos\alpha\cos\beta\cos\gamma$, $\alpha+\beta+\gamma$, and similar expressions. These four points are all on the line whose equation is

$$2a\cos\alpha\cos\beta\cos\gamma\cos\delta=r\cos(\theta-\alpha-\beta-\gamma-\delta).$$

This proposition can clearly be extended.

EXAMPLES ON CHAPTER IV.

1. A point moves so that the square of its distance from a fixed point varies as its perpendicular distance from a fixed straight line; shew that it describes a circle.

2. A point moves so that the sum of the squares of its distances from the four sides of a square is constant; shew that the locus of the point is a circle.

3. The locus of a point, the sum of the squares of whose distances from n fixed points is constant, is a circle.

4. A, B are two fixed points, and P moves so that $PA = n \,.\, PB$; shew that the locus of P is a circle. Shew also that, for different values of n, all the circles have a common radical axis.

5. Find the locus of a point which moves so that the square of its distance from the base of an isosceles triangle is equal to the rectangle under its distances from the other sides.

6. Prove that the equation of the circle circumscribing the triangle formed by the lines $x + y = 6$, $2x + y = 4$, and $x + 2y = 5$ is

$$x^2 + y^2 - 17x - 19y + 50 = 0.$$

7. Find the equation of the circle whose diameter is the common chord of the circles

$$x^2 + y^2 + 2x + 3y + 1 = 0, \text{ and } x^2 + y^2 + 4x + 3y + 2 = 0.$$

8. Find the equation of the straight lines joining the origin to the points of intersection of the line $x + 2y - 3 = 0$, and the circle $x^2 + y^2 - 2x - 2y = 0$, and shew that the lines are at right angles to one another.

9. Any straight line is drawn from a fixed point O meeting a fixed straight line in P, and a point Q is taken on the line such that the rectangle $OQ \,.\, OP$ is constant; shew that the locus of Q is a circle.

10. Any straight line is drawn from a fixed point O meeting a fixed circle in P, and a point Q is taken on the line such that the rectangle $OQ \,.\, OP$ is constant; shew that the locus of Q is a circle.

11. Shew that the radical axis of two circles bisects their four common tangents.

12. If O be one of the limiting points of a system of co-axial circles, shew that a common tangent to any two circles of the system will subtend a right angle at O.

13. Prove that the equation of two given circles can always be put in the form

$$x^2 + y^2 + ax + b = 0, \quad x^2 + y^2 + a'x + b = 0,$$

and that one of the circles will be within the other if aa' and b are both positive.

14. The distances of two points from the centre of a circle are proportional to the distances of each from the polar of the other.

15. If a circle be described on the line joining the centres of similitude of two given circles as diameter, prove that the tangents drawn from any point on it to the two circles are in the ratio of the corresponding radii.

16. Find the locus of a point which is such that tangents from it to two concentric circles are inversely as their radii.

17. If two points A, B are harmonic conjugates with respect to two others C, D, the circles on AB and CD as diameters cut orthogonally.

18. If two circles cut orthogonally, every diameter of either which meets the other is cut harmonically.

19. A point moves so that the sum of the squares of its distances from the sides of a regular polygon is constant; shew that its locus is a circle.

20. A circle passes through a fixed point O and cuts two fixed straight lines through O, which are at right angles to one another, in points P, Q, such that the line PQ always passes through a fixed point; find the equation of the locus of the centre of the circle.

21. The polar equation of the circle on (a, α), (b, β) as diameter is

$$r^2 - r\{a\cos(\theta-\alpha) + b\cos(\theta-\beta)\} + ab\cos(\alpha-\beta) = 0.$$

22. Find the equation for determining the values of r at the points of intersection of the circle and the straight line whose equations are

$$r = 2a\cos\theta, \text{ and } r\cos(\theta-\beta) = p.$$

Deduce the value of p when the straight line becomes a tangent.

23. Find the co-ordinates of the centre of the inscribed circle of the triangle the equations of whose sides are

$$3x - 4y = 0,\ 7x - 24y = 0, \text{ and } 5x - 12y - 36 = 0.$$

24. Find the locus of a point the polars of which with respect to two given circles make a given angle with one another.

25. From any point on the radical axis of two circles tangents are drawn, and the lines joining the points of contact to the centres of the circles are produced to meet; find the equation of the locus of the point of intersection.

26. If the four points in which the two circles

$$x^2+y^2+ax+by+c=0, \quad x^2+y^2+a'x+b'y+c'=0$$

are intersected by the straight lines

$$Ax+By+C=0, \; A'x+B'y+C'=0$$

respectively, lie on another circle, then will

$$\begin{vmatrix} a-a', & b-b', & c-c' \\ A, & B, & C \\ A', & B', & C' \end{vmatrix}=0.$$

27. A system of circles is drawn through two fixed points, tangents are drawn to these circles parallel to a given straight line; find the equation of the locus of the points of contact.

28. If A, B, C be the centres of three co-axial circles, and t_1, t_2, t_3 be the tangents to them from any point, prove the relation

$$BCt_1^2+CAt_2^2+ABt_3^2=0.$$

29. If t_1, t_2, t_3 be the lengths of the tangents from any point to three given circles, whose centres are not in the same straight line, shew that any circle or any straight line can be represented by an equation of the form

$$At_1^2+Bt_2^2+Ct_3^2=D.$$

What relation will hold between A, B, C for straight lines?

30. If a circle cut two of a system of co-axial circles at right angles, it will cut them all at right angles.

31. Shew that every circle which passes through two given points is cut orthogonally by each of a system of circles having a common radical axis.

32. Prove that all circles touching two fixed circles are orthogonal to one of two other fixed circles.

33. If two circles cut orthogonally, prove that an indefinite number of pairs of points can be found on their common diameter such that either point has the same polar with respect to one circle that the other has with respect to the other. Also shew that the distance between such pairs of points subtends a right angle at one of the points of intersection of the two circles.

34. If the equations of two circles whose radii are a, a' be $S=0$, $S'=0$, then the circles

$$\frac{S}{a} \mp \frac{S'}{a'} = 0$$

will intersect at right angles.

35. Find the locus of the point of intersection of two straight lines at right angles to one another, each of which touches one of the two circles

$$(x-a)^2 + y^2 = b^2, \quad (x+a)^2 + y^2 = c^2,$$

and prove that the bisectors of the angles between the straight lines always touch one or other of two other fixed circles.

36. Shew that the diameter of the circle which cuts at right angles the three escribed circles of the triangle ABC is

$$\frac{a}{\sin A}\left(1 + \cos A \cos B + \cos B \cos C + \cos C \cos A\right)^{\frac{1}{2}}.$$

37. Find the locus of the point of contact of two equal circles of constant radius c, each of which passes through one of two fixed points at a distance $2a$ apart: and shew that, if $a=c$, the locus splits up into a circle of radius a and a curve whose equation may be put into the form $(x^2+y^2)^2 = a^2(x^2-3y^2)$.

89-94.

CHAPTER V.

THE PARABOLA.

89. *Definitions.* A *Conic Section*, or *Conic*, is the locus of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed straight line. The fixed point is called a *focus*, the fixed straight line is called a *directrix*, and the constant ratio is called the *eccentricity*.

It will be shewn hereafter [Art. 312] that if a right circular cone be cut by any plane, the section will be in all cases a conic as defined above. It was as sections of a cone that the properties of these curves were first investigated.

We proceed to find the equation and discuss some of the properties of the simplest of these curves, namely that in which the eccentricity is equal to *unity*. This curve is called a *parabola*.

90. *To find the equation of a parabola.*

Let S be the focus, and let YY' be the directrix. Draw SO perpendicular to YY', and let $OS = 2a$. Take OS for the axis of x, and OY for the axis of y.

Let P be any point on the curve, and let the coordinates of P be x, y.

Draw PN, PM, perpendicular to the axes, as in the figure, and join SP.

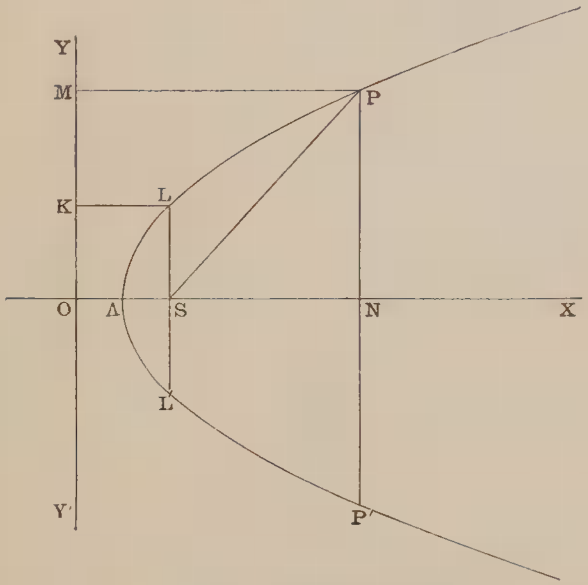
Then, by definition, $SP = PM$;

therefore $$PM^2 = SP^2 = PN^2 + SN^2;$$

that is, $$x^2 = y^2 + (x - 2a)^2,$$

or $$y^2 = 4a(x - a) \quad\text{(i)}.$$

This is the required equation of the curve.



The curve cuts the axis of x at a point A where $y = 0$ and from (i) when $y = 0$, $x = a$; that is, $OA = a$.

The point A is called the *vertex* of the parabola.

If we transfer the origin to A, the axes being unchanged in direction, equation (i) will become [Art. 49]

$$y^2 = 4ax \quad\text{(ii)}.$$

The focus is the point $(a, 0)$. The directrix is the line

$$x + a = 0.$$

Also $$SP = MP = OA + AN = a + x.$$

91. Since the equation of the parabola is $y^2 = 4ax$, and y^2 is a positive quantity, x must always be positive,

and therefore the curve lies wholly on the positive side of the axis of y.

For any particular value of x there are clearly two values of y equal in magnitude, one being positive and the other negative. Hence all chords of the curve perpendicular to the axis of x are bisected by it, and the portions of the curve on the positive and on the negative sides of the axis of x are in all respects equal.

As x increases y also increases, and there is no limit to this increase of x and y, so that there is no limit to the curve on the positive side of the axis of y.

The line through the focus perpendicular to the directrix is called the *axis* of the parabola.

The chord through the focus perpendicular to the axis is called the *latus-rectum.*

In the figure to Art. 90, $SL = LK = OS = 2a$. Therefore the whole length of the latus-rectum is $4a$.

92. We have found that $y^2 - 4ax = 0$ for all points on the parabola.

For all points *within* the curve $y^2 - 4ax$ is negative.

For, if Q be such a point, and through Q a line be drawn perpendicular to the axis meeting the curve in P and the axis in N, then Q is nearer to the axis than P and therefore NQ^2 is less than NP^2. But, P being on the curve, $NP^2 - 4a \, . \, AN = 0$, and therefore $NQ^2 - 4a \, . \, AN$ is *negative.*

Similarly we may prove that for all points *outside* the curve $y^2 - 4ax$ is positive.

Hence, if the equation of a parabola be $y^2 - 4ax = 0$, and we substitute the co-ordinates of any point in the left-hand member of the equation, the result will be positive if the point be outside the curve, negative if the point be within the curve, and zero if the point be upon the curve.

93. The co-ordinates of the points common to the straight line, whose equation is $y = mx + c$, and the

parabola, whose equation is $y^2 = 4ax$, must satisfy both equations.

Hence, at a common point, we have the relation,

$$(mx + c)^2 = 4ax \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Therefore the abscissæ of the common points are given by the equation (i), which may be written in the form

$$m^2x^2 + (2mc - 4a)\,x + c^2 = 0 \ldots\ldots\ldots\ldots\text{(ii)}.$$

Since (ii) is a quadratic equation, we see that every straight line meets a parabola in *two* points, which may be real, coincident, or imaginary.

When m is very small, one root of the equation (ii) is very great; when m is equal to zero, one root is infinitely great. Hence every straight line parallel to the axis of a parabola meets the curve in one point at a finite distance, and in another at an infinite distance from the vertex.

94. *To find the condition that the line* $y = mx + c$ *may touch the parabola* $y^2 - 4ax = 0$.

As in the preceding Article, the abscissæ of the points common to the straight line and the parabola are given by the equation

$$(mx + c)^2 = 4ax,$$

that is $$m^2x^2 + (2mc - 4a)\,x + c^2 = 0.$$

If the line be a tangent, that is if it cut the parabola in two coincident points, the roots of the equation must be equal to one another. The condition for this is

$$4m^2c^2 = (2mc - 4a)^2,$$

which reduces to $mc = a$, or $c = \dfrac{a}{m}$.

Hence, whatever m may be, the line

$$y = mx + \frac{a}{m}$$

will touch the parabola $y^2 - 4ax = 0$.

95. *To find the equation of the straight line passing through two given points on a parabola, and to find the equation of the tangent at any point.*

Let the equation of the parabola be

$$y^2 = 4ax,$$

and let x', y', and x'', y'' be the co-ordinates of two points on it.

The equation of the line through these points is

$$\frac{y - y'}{y'' - y'} = \frac{x - x'}{x'' - x'} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

But since the points are on the parabola, we have

$$y'^2 = 4ax', \text{ and } y''^2 = 4ax'';$$

$$\therefore\; y''^2 - y'^2 = 4a\,(x'' - x') \ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

By multiplying the corresponding sides of the equations (i) and (ii), we have

$$(y - y')\,(y'' + y') = (x - x')\,4a,$$

or, since $y'^2 - 4ax' = 0$,

$$y\,(y' + y'') - 4ax - y'y'' = 0 \;\ldots\ldots\ldots\text{(iii)},$$

which is the equation of the chord joining the two given points.

In order to find the equation of the tangent at (x', y') we must put $y'' = y'$ and $x'' = x'$ in equation (iii), and we obtain

$$2yy' - 4ax - y'^2 = 0,$$

or, since $y'^2 = 4ax'$,

$$yy' = 2a\,(x + x') \;\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}.$$

Cor. The tangent at (0, 0) is $x = 0$; that is, the tangent at the vertex is perpendicular to the axis.

96. We have found by independent methods [Articles 94 and 95] two forms of the equation of a tangent to a parabola. Either of these could however have been found from the other. Thus, suppose we know that the equation of the tangent at (x', y') is

$$yy' = 2a\,(x + x'),$$

then

$$y = \frac{2a}{y'}\,x + \frac{2ax'}{y'}.$$

If this be the same line as that given by

$$y = mx + c$$

we must have

$$m = \frac{2a}{y'}, \text{ and } c = \frac{2ax'}{y'};$$

therefore $mc = a$, as in Article 94.

In the solution of questions we should take whichever form of the equation of a tangent appears the more suitable for the particular case.

Ex. 1. *The ordinate of the point of intersection of two tangents to a parabola is the arithmetic mean between the ordinates of the points of contact of the tangents.*

The equations of the tangents at the points (x', y') and (x'', y'') are

$$yy' = 2a(x + x'),$$

and

$$yy'' = 2a(x + x'').$$

By subtraction, we have for their common point,

$$\begin{aligned} y(y' - y'') &= 2ax' - 2ax'' \\ &= \tfrac{1}{2}(y'^2 - y''^2); \\ \therefore\ y &= \tfrac{1}{2}(y' + y''). \end{aligned}$$

Ex. 2. *To find the locus of the point of intersection of two tangents to a parabola which are at right angles to one another.*

Let the equations of the two tangents be

$$y = mx + \frac{a}{m} \quad \text{.............................(i)},$$

$$y = m'x + \frac{a}{m'} \quad \text{.............................(ii)}.$$

Then, since they are at right angles, $mm' = -1$. Hence the second equation can be written,

$$y = -\frac{1}{m}x - am \quad \text{...........................(iii)}.$$

To find the abscissa of their common point we have only to subtract (iii) from (i), and we get

$$0 = x\left(m + \frac{1}{m}\right) + a\left(m + \frac{1}{m}\right);$$

and therefore we have $x + a = 0$.

The equation of the required locus is therefore $x + a = 0$, and this [Art. 90] is the equation of the directrix.

97. *To find the equation of the normal at any point of a parabola.*

The equation of the tangent at (x', y') to the parabola $y^2 - 4ax = 0$, is [Art. 95]

$$yy' = 2a(x + x') \quad \text{..................(i)}.$$

The normal is the perpendicular line through (x', y').

Therefore [Art. 30] its equation is

$$(y - y')\,2a + y'(x - x') = 0 \quad \text{...... ...(ii)}.$$

The above equation may be written

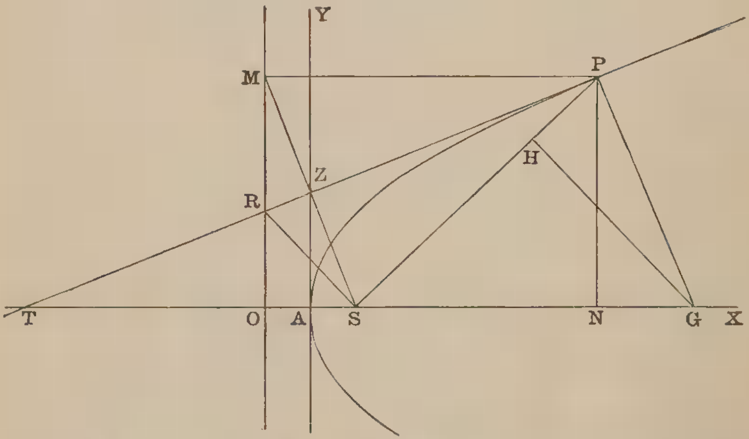
$$y = -\frac{y'}{2a}x + y' + \frac{y'^3}{8a^2} \ldots\ldots\ldots\ldots\text{(iii)}.$$

If we put $m = -\frac{y'}{2a}$, then $y' = -2am$, and $\frac{y'^3}{8a^2} = -am^3$; therefore (iii) becomes

$$y = mx - 2am - am^3 \ldots\ldots\ldots\ldots\text{(iv)}.$$

This form of the equation of a normal is sometimes useful.

98. We will now prove some geometrical properties of a parabola.



Let the tangent at the point P meet the directrix in R and the axis in T. Let PN, PM be the perpendiculars from P on the axis and on the directrix.

Let PG, the normal at P, meet the axis in G.

Then, if x', y' be the co-ordinates of P, the equation of the tangent at P will be

$$yy' = 2a(x + x') \ldots\ldots\text{(i)} \quad \text{[Art. 95]}.$$

Where this meets the axis, $y = 0$, and at that point, we have from (i), $x + x' = 0$.

$$\therefore \quad TA = AN \ldots\ldots\ldots\ldots\ldots(\alpha);$$

$$\therefore \quad TS = AS + AN = SP \ldots\ldots\ldots\ldots(\beta);$$

and since $TS = SP$, the angle STP is equal to the angle SPT; so that PT bisects the angle SPM............(γ).

We see also that the triangles RSP and RMP are equal in all respects.

Hence the angle RSP = the angle RMP = a right angle ... (δ).

Again, since M is the point $(-a, y')$, and S is the point $(a, 0)$, the equation of the line SM is

$$\frac{y-y'}{-y'} = \frac{x+a}{2a} \text{....................(ii).}$$

This is clearly perpendicular [Art. 30] to the tangent at P which is given by the equation (i),

$\therefore$ SM is perpendicular to PT.........(ϵ).

Since PT is perpendicular to SM and bisects the angle SPM, it will bisect SM. If then Z be the point of intersection of SM and PT, $SZ = ZM$. But $SA = AO$. Therefore AZ is parallel to OM, and is therefore the tangent at the vertex of the parabola; so that the line through the focus of a parabola perpendicular to any tangent PT meets PT on the tangent at the vertex..............(ζ).

We may prove the last proposition as follows.

Let the equation of any tangent to the parabola be

$$y = mx + \frac{a}{m} \text{.. ....(iii).}$$

The equation of the line through the focus $(a, 0)$ perpendicular to (iii) is

$$y = -\frac{1}{m}(x-a),$$

or

$$y = -\frac{x}{m} + \frac{a}{m} \text{ ....(iv).}$$

The lines (iii) and (iv) clearly meet where $x = 0$.

The equation of the normal at P (x', y') is [Art. 97]

$$2a(y-y') + y'(x-x') = 0.$$

At the point G we have $y = 0$, and therefore

$$-2ay' + y'(x-x') = 0,$$

or
$$2a = x - x' = AG - AN = NG.$$
$$\therefore\ NG = 2a \ldots\ldots\ldots\ldots\ldots\ldots (\eta).$$

EXAMPLES.

1. Find the equations of the tangents and the equations of the normals to the parabola $y^2 - 4ax = 0$ at the ends of its latus rectum.

Ans. $x \mp y + a = 0,\quad y \pm x \mp 3a = 0.$

2. Find the points where the line $y = 3x - a$ cuts the parabola $y^2 - 4ax = 0$.

Ans. $(a,\ 2a),\ \left(\frac{a}{9},\ -\frac{2}{3}a\right).$

3. Shew that the tangent to the parabola $y^2 - 4ax = 0$ at the point $(x',\ y')$ is perpendicular to the tangent at the point
$$\left(\frac{a^2}{x'},\ \frac{-4a^2}{y'}\right).$$

4. Shew that the line $y = 2x + \frac{a}{2}$ cuts $y^2 - 4ax = 0$ in coincident points. Shew that it also cuts $20x^2 + 20y^2 = a^2$ in coincident points.

5. A straight line touches both $x^2 + y^2 = 2a^2$ and $y^2 = 8ax$; shew that its equation is $y = \pm(x + 2a)$.

6. Shew that the line $7x + 6y = 13$ is a tangent to the curve
$$y^2 - 7x - 8y + 14 = 0.$$

7. Shew that the equation $x^2 + 4ax + 2ay = 0$ represents a parabola, whose vertex is at the point $(-2a,\ 2a)$, whose latus rectum is $2a$, and whose axis is parallel to the axis of y.

8. Shew that all parabolas whose axes are parallel to the axis of y have their equations of the form
$$x^2 + 2Ax + 2By + C = 0.$$

9. Find the co-ordinates of the vertex and the length of the latus rectum of each of the following parabolas:

(i) $y^2 = 5x + 10$, (ii) $x^2 - 4x + 2y = 0$,
(iii) $(y - 2)^2 = 5(x + 4)$, and (iv) $3x^2 + 12x - 8y = 0$.

Ans. (i) $(-2,\ 0),\ 5$. (ii) $(2,\ 2),\ 2$. (iii) $(-4,\ 2),\ 5$. (iv) $(-2,\ -\frac{3}{2}),\ \frac{8}{3}$.

10. Find the co-ordinates of the focus and the equation of the directrix of each of the parabolas in question 9.

Ans. (i) $(-\frac{3}{4},\ 0),\ 4x + 13 = 0$. (ii) $(2,\ \frac{3}{2}),\ 2y - 5 = 0$.
(iii) $(-\frac{11}{4},\ 2),\ 4x + 21 = 0$. (iv) $(-2,\ -\frac{5}{6}),\ 6y + 13 = 0$.

11. Write down the equation of the parabola whose focus is the origin and directrix the straight line $2x - y - 1 = 0$. Shew that the line $2y = 4x - 1$ touches the parabola.

12. If through a fixed point O on the axis of a parabola any chord POP' be drawn, shew that the rectangle of the ordinates of P and P' will be constant. Shew also that the product of the abscissæ will be constant.

13. Find the co-ordinates of the point of intersection of the tangents $y=mx+\frac{a}{m}$, $y=m'x+\frac{a}{m'}$. Shew that the locus of their intersection is a straight line whenever mm' is constant; and that, when $mm'+1=0$, this line is the directrix. Omit.

14. Shew that, for all values of m, the line $y=m(x+a)+\frac{a}{m}$ will touch the parabola $y^2=4a(x+a)$.

15. Two lines are at right angles to one another, and one of them touches $y^2=4a(x+a)$, and the other $y^2=4a'(x+a')$; shew that the point of intersection of the lines will be on the line $x+a+a'=0$.

16. If perpendiculars be let fall on any tangent to a parabola from two given points on the axis equidistant from the focus, the difference of their squares will be constant.

17. Two straight lines AP, AQ are drawn through the vertex of a parabola at right angles to one another, meeting the curve in P, Q; shew that the line PQ cuts the axis in a fixed point.

18. If the circle $x^2+y^2+Ax+By+C=0$ cut the parabola $y^2-4ax=0$ in four points, the algebraic sum of the ordinates of those points will be zero.

19. If the tangent to the parabola $y^2-4ax=0$ meet the axis in T and the tangent at the vertex A in Y, and the rectangle $TAYQ$ be completed; shew that the locus of Q is the parabola $y^2+ax=0$.

20. If P, Q, R be three points on a parabola whose ordinates are in geometrical progression, shew that the tangents at P, R will meet on the ordinate of Q.

21. Shew that the area of the triangle inscribed in the parabola $y^2-4ax=0$ is $\frac{1}{8a}(y_1\sim y_2)\,(y_2\sim y_3)\,(y_3\sim y_1)$, where y_1, y_2, y_3 are the ordinates of the angular points.

99. *Two tangents can be drawn to a parabola from any point, which will be real, coincident, or imaginary, according as the point is outside, upon, or within the curve.*

99-103

The line whose equation is

$$y = mx + \frac{a}{m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

will touch the parabola $y^2 = 4ax$, whatever the value of m may be [Art. 94].

The line (i) will pass through the particular point (x', y'), if

$$y' = mx' + \frac{a}{m},$$

that is if $$m^2x' - my' + a = 0 \ldots\ldots\ldots\ldots\text{(ii)}.$$

Equation (ii) is a quadratic equation which gives the directions of those tangents to the parabola which pass through the point (x', y'). Since a quadratic equation has *two* roots, *two* tangents will pass through any point (x', y').

The roots of (ii) are real, coincident, or imaginary, according as $y'^2 - 4ax'$ is positive, zero, or negative. That is [Art. 92] according as (x', y') is outside the parabola, upon the parabola, or within it.

100. *To find the equation of the line through the points of contact of the two tangents which can be drawn to a parabola from any point.*

Let x', y' be the co-ordinates of the point from which the tangents are drawn.

Let the co-ordinates of the points of contact of the tangents be h, k and h', k' respectively.

The equations of the tangents at (h, k) and (h', k') are

$$yk = 2a(x + h)$$

and $$yk' = 2a(x + h').$$

We know that (x', y') is on both these lines;

$$\therefore\ y'k = 2a(x' + h) \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

and $$y'k' = 2a(x' + h') \ldots\ldots\ldots\ldots\text{(ii)}.$$

But the equations (i) and (ii) are the conditions that the points (h, k) and (h', k') may lie on the straight line whose equation is

$$y'y = 2a(x' + x) \ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Hence (iii) is the required equation of the line through the points of contact of the tangents from (x', y').

The line joining the points of contact of the two tangents from any point P to a parabola is called the *polar* of P with respect to the parabola. [See Art. 76.]

101. *If the polar of the point P with respect to a parabola pass through the point Q, then will the polar of Q pass through P.*

Let the co-ordinates of P be x', y', and the co-ordinates of Q be x'', y''.

The equation of the polar of P with respect to the parabola $y^2 - 4ax = 0$ is

$$yy' = 2a(x + x').$$

If this line pass through Q (x'', y''), we must have

$$y''y' = 2a(x'' + x').$$

The symmetry of this result shews that it is also the condition that the polar of Q should pass through P.

It can be shewn, exactly as in Art. 79, that if the polars of two points P, Q meet in R, then R is the pole of the line PQ.

The equation of the polar of the focus $(a, 0)$ is $x + a = 0$. So that the polar of the focus is the directrix.

If Q be any point on the directrix, Q is on the polar of the focus S, therefore the polar of Q will pass through S, so that if tangents be drawn to a parabola from any point on the directrix the line joining the points of contact will pass through the focus.

102. *The locus of the middle points of a system of parallel chords of a parabola is a straight line parallel to the axis of the parabola.*

The equation of the straight line joining the two points (x', y'), (x'', y'') on the parabola $y^2 - 4ax = 0$ is [Art. 95, (iii)]

$$y(y' + y'') - 4ax - y'y'' = 0 \text{ ....(i).}$$

Now, if the line (i) make an angle θ with the axis of the parabola,

$$\tan\theta = \frac{4a}{y' + y''}\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

But, if the co-ordinates of the middle point of the chord be (x, y), then will

$$2x = x' + x'', \text{ and } 2y = y' + y''.$$

Hence, from (ii), $$\tan\theta = \frac{4a}{2y},$$

or $$y = 2a\cot\theta\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)},$$

so that y is constant so long as θ is constant.

Hence the locus of the middle points of a system of parallel chords of a parabola is a straight line parallel to the axis of the parabola.

Def. The locus of the middle points of a system of parallel chords of a conic is called a *diameter*, and the chords it bisects are called the ordinates of that diameter.

We have seen in Art. 93 that a diameter of a parabola only meets the curve in *one* point at a finite distance from the vertex. The point where a diameter cuts the curve is called the extremity of that diameter.

103. *The tangent at the extremity of a diameter is parallel to the chords which are bisected by that diameter.*

All the middle points of a system of parallel chords of a parabola are on a diameter. Hence, by considering the parallel tangent, that is the parallel chord which cuts the curve in coincident points, we see that the diameter of a system of parallel chords passes through the point of contact of the tangent which is parallel to the chords.

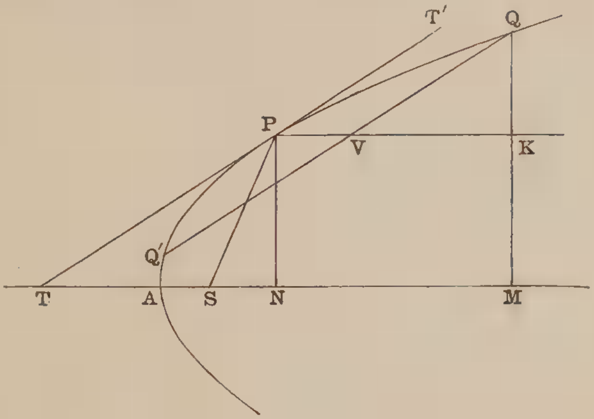
104. *To find the equation of a parabola when the axes are any diameter and the tangent at the extremity of that diameter.*

Let P be the extremity of the diameter, and let the tangent at P make an angle θ with the axis.

Then $$NP = 2a\cot\theta \text{ [Art. 102 (iii)]},$$

$$\therefore\ AN = \frac{PN^2}{4a} = a\cot^2\theta.$$

Let the co-ordinates of Q referred to the new axes be x, y respectively, and draw QM perpendicular to the axis of the parabola, cutting the diameter PV in K.



Then

$$MQ = NP + KQ = 2a \cot\theta + y \sin\theta \ldots\ldots \text{(i)},$$

$$AM = AN + NM = AN + PV + VK$$

$$= a \cot^2\theta + x + y \cos\theta \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

But $$QM^2 = 4a \,.\, AM;$$

therefore, from (i) and (ii),

$$(2a \cot\theta + y \sin\theta)^2 = 4a\,(a \cot^2\theta + x + y\cos\theta),$$

or $$y^2 \sin^2\theta = 4ax \ldots\ldots\ldots\ldots\ldots \text{(iii)}.$$

But $AN = a \cot^2\theta$; therefore $SP = a + AN = \dfrac{a}{\sin^2\theta}$.

Therefore, putting a' for SP or $\dfrac{a}{\sin^2\theta}$, the equation of the curve is $$y^2 = 4a'x \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}.$$

105. If the equation of a parabola, referred to any diameter and the tangent at the extremity of that diameter as axes, be $y^2 - 4ax = 0$; the line $y = mx + \dfrac{a}{m}$ will be a tangent for all values of m; the equation of the tangent

at any point (x', y') will be $yy' - 2a(x + x') = 0$; the equation of the polar of (x', y') with respect to the parabola will be $yy' - 2a(x + x') = 0$; and the locus of the middle points of chords parallel to the line $y = mx$ will be $y = \frac{2a}{m}$.

These propositions require no fresh investigations; for Articles 94, 95, 100 and 102 hold good equally whether the axes are at right angles or not.

106. The equation of the normal at any point (x', y') of the parabola $y^2 - 4ax = 0$ is

$$y - y' + \frac{y'}{2a}(x - x') = 0 \text{............(i).}$$

If the line (i) pass through the point (h, k) we have

$$k - y' + \frac{y'}{2a}\left(h - \frac{y'^2}{4a}\right) = 0 \text{..... ......(ii).}$$

The equation (ii) gives the ordinates of the points the normals at which pass through the particular point (h, k). The equation is a *cubic* equation, and therefore through any point *three* normals can be drawn to a parabola.

Since there is no term containing y'^2, we have, if y_1, y_2, y_3 be the three roots of the equation (ii),

$$y_1 + y_2 + y_3 = 0 \text{...............(iii).}$$

Now, for a system of parallel chords of a parabola, the sum of the two ordinates at the ends of any chord is constant [Art. 102]. Therefore the normals at these points meet on the normal at a fixed point the ordinate of which added to the sum of their ordinates is zero.

Hence the locus of the intersection of the normals at the ends of a system of parallel chords of a parabola is a straight line which is a normal to the curve.

107. We shall conclude this Chapter by the solution of some examples.

(1) *To find the locus of the point of intersection of two tangents to a parabola which make a given angle with one another.*

The line $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 - 4ax = 0$, whatever the value of m may be. [Art. 94.]

If (x, y) be supposed known, the equation will give the directions of the tangents which pass through that point.

The equation giving the directions will be

$$m^2x - my + a = 0.$$

And, if the roots of this quadratic equation be m_1 and m_2, then will

$$m_1 + m_2 = \frac{y}{x} \text{ and } m_1m_2 = \frac{a}{x};$$

$$\therefore (m_1 - m_2)^2 = \frac{y^2 - 4ax}{x^2}.$$

But, if the two tangents make an angle α with one another,

$$\tan\alpha = \frac{m_1 - m_2}{1 + m_1m_2}$$

$$\frac{\sqrt{\frac{y^2}{x^2} - \frac{4a}{x}}}{1 + a}$$

$$\therefore \tan^2\alpha = \frac{y^2 - 4ax}{(a + x)^2}.$$

So that the equation of the required locus is

$$y^2 - 4ax - (x + a)^2 \tan^2\alpha = 0.$$

(2) *To find the locus of the foot of the perpendicular drawn from a fixed point to any tangent to a parabola.*

Let the equation of the parabola be $y^2 - 4ax = 0$, and let h, k be the co-ordinates of the fixed point O.

The equation of any tangent to the parabola is

$$y = mx + \frac{a}{m} \quad \text{..............................} \quad \text{(i)}.$$

The equation of a line through (h, k) perpendicular to (i) is

$$y - k = -\frac{1}{m}(x - h) \quad \text{..............................} \quad \text{(ii)}.$$

To find the locus we have to eliminate m between the equations (i) and (ii).

From (ii) we have $\quad m = -\frac{x - h}{y - k}$;

therefore, by substituting in (i), we get

$$y + \frac{x - h}{y - k}x + a\frac{y - k}{x - h} = 0,$$

or $\quad y(y - k)(x - h) + x(x - h)^2 + a(y - k)^2 = 0 \quad \text{..............} \quad \text{(iii)}.$

The locus is therefore a curve of the third degree.

From (iii) we see that the point O itself is always on the locus. If the point O be outside the parabola this presents no difficulty, for two real tangents can in that case be drawn through O, and the feet of the perpendiculars from O on these will be O itself. When the point O is within the parabola the tangents from O are imaginary, and the perpendiculars to them from O are also imaginary, but they all pass through the real point O, and therefore O is a point on the locus.

When $h=a$, $k=0$, that is when O is the focus of the parabola, (iii) reduces to $x\{y^2+(x-a)^2\}=0$; so that the cubic reduces to the point circle $y^2+(x-a)^2=0$, and the straight line $x=0$. [See Art. 98 ζ].

(3) *The orthocentre of the triangle formed by three tangents to a parabola is on the directrix.*

Let the equations of the sides of the triangle be

$$y=m'x+\frac{a}{m'},\ y=m''x+\frac{a}{m''},\ \text{and}\ y=m'''x+\frac{a}{m'''}.$$

The point of intersection of the second and third sides is

$$\left(\frac{a}{m''m'''},\ \frac{a}{m''}+\frac{a}{m'''}\right).$$

The line through this point perpendicular to the first side has for equation

$$y-\frac{a}{m''}-\frac{a}{m'''}=-\frac{1}{m'}\left(x-\frac{a}{m''m'''}\right).$$

Now this line cuts the directrix, whose equation is $x=-a$, in the point whose ordinate is equal to

$$a\left(\frac{1}{m'}+\frac{1}{m''}+\frac{1}{m'''}+\frac{1}{m'm''m'''}\right).$$

The symmetry of this result shews that the other perpendiculars cut the directrix in the same point, which proves the theorem.

(4) *To find the locus of the point of intersection of two normals which are at right angles to one another.*

The line whose equation is

$$y=mx-2am-am^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

is a normal to the parabola $y^2-4ax=0$, whatever the value of m may be.

If the point (x, y) be supposed known, the equation (i) gives the directions of the normals which pass through that point.

If the roots of the equation (i) be m_1, m_2, m_3, we have

$$m_1+m_2+m_3=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)},$$

$$m_1m_2+m_2m_3+m_3m_1=\frac{2a-x}{a} \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)},$$

$$m_1m_2m_3=-\frac{y}{a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}.$$

If two of the normals, given by m_1, m_2 suppose, be perpendicular to one another, we have

$$m_1 m_2 = -1 \quad \text{.........................} \quad \text{(v).}$$

The elimination of m_1, m_2, m_3 from the equations (ii), (iii), (iv) and (v) will give the locus required.

The result is $$y^2 = a(x - 3a).$$

EXAMPLES ON CHAPTER V.

1. The perpendicular from a point O on its polar with respect to a parabola meets the polar in the point M and cuts the axis in G, the polar cuts the axis in T, and the ordinate through O cuts the curve in P, P'; shew that the points T, P, M, G, P' are all on a circle whose centre is S.

2. Prove that the two parabolas $y^2 = ax$, $x^2 = by$ will cut one another at an angle

$$\tan^{-1} \frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2(a^{\frac{2}{3}} + b^{\frac{2}{3}})}.$$

3. If PSQ be a focal chord of a parabola, and PA meet the directrix in M, shew that MQ will be parallel to the axis of the parabola.

4. Shew that the locus of the point of intersection of two tangents to a parabola at points on the curve whose ordinates are in a constant ratio is a parabola.

5. The two tangents from a point P to the parabola $y^2 - 4ax = 0$ make angles θ_1, θ_2 with the axis of x; find the locus of P (i) when $\tan\theta_1 + \tan\theta_2$ is constant, and (ii) when $\tan^2\theta_1 + \tan^2\theta_2$ is constant.

6. Find the equation of the locus of the point of intersection of two tangents to a parabola which make an angle of 45° with one another.

7. Shew that if two tangents to a parabola intercept a fixed length on the tangent at the vertex, the locus of their intersection is another equal parabola.

8. Shew that two tangents to a parabola which make equal angles respectively with the axis and directrix but are not at right angles, intersect on the latus rectum.

9. From any point on the latus rectum of a parabola perpendiculars are drawn to the tangents at its extremities; shew that the line joining the feet of these perpendiculars touches the parabola.

10. Shew that if tangents be drawn to the parabola $y^2 - 4ax = 0$ from a point on the line $x + 4a = 0$, their chord of contact will subtend a right angle at the vertex.

11. The perpendicular TN from any point T on its polar with respect to a parabola meets the axis in M; shew that if $TN \,.\, TM$ is constant the locus of T is a parabola; shew also that if the ratio $TN : TM$ is constant the locus is a parabola.

12. Two equal parabolas have their axes parallel and a common tangent at their vertices: straight lines are drawn parallel to the direction of either axis; shew that the locus of the middle points of the parts of the lines intercepted between the curves is an equal parabola.

13. Two parabolas touch one another and have their axes parallel; shew that, if the tangents at two points of these parabolas intersect in any point on their common tangent, the line joining their points of contact will be parallel to the axis.

14. Two parabolas have the same axis; tangents are drawn from points on the first to the second; prove that the middle points of the chords of contact with the second lie on a fixed parabola.

15. Shew that the locus of the middle point of a chord of a parabola which passes through a fixed point is a parabola.

16. The middle point of a chord PP' is on a fixed straight line perpendicular to the axis of a parabola; shew that the locus of the pole of the chord is another parabola.

17. If TP, TQ be tangents to a parabola whose vertex is A, and if the lines AP, AT, AQ, produced if necessary, cut the directrix in p, t, and q respectively; shew that $pt = tq$.

18. If the diameter through any point O of a parabola meet any chord in P, and the tangents at the ends of that chord meet the diameter in Q, Q'; shew that $OP^2 = OQ \cdot OQ'$.

19. The vertex of a triangle is fixed, the base is of constant length and moves along a fixed straight line; shew that the locus of the centre of its circumscribing circle is a parabola.

20. Shew that the polar of any point on the circle

$$x^2 + y^2 - 2ax - 3a^2 = 0,$$

with respect to the circle

$$x^2 + y^2 + 2ax - 3a^2 = 0,$$

will touch the parabola

$$y^2 + 4ax = 0.$$

21. PSP' is a focal chord of a parabola; V is the middle point of PP', and VO is perpendicular to PP' and cuts the axis in O; prove that SO, VO are the arithmetic and geometric means between SP' and SP.

22. PSp, QSq, RSr are three focal chords of a parabola; QR meets the diameter through p in A, RP meets the diameter through q in B, and PQ meets the diameter through r in C; shew that the three points A, B, C are on a straight line through S.

23. PP' is any one of a system of parallel chords of a parabola, O is a point on PP' such that the rectangle $PO \cdot OP'$ is constant; shew that the locus of O is a parabola.

24. On the diameter through a point O of a parabola two points P, P' are taken so that $OP \cdot OP'$ is constant; prove that the four points of intersection of the tangents drawn from P, P' to the parabola will lie on two fixed straight lines parallel to the tangent at O and equidistant from it.

25. If a quadrilateral circumscribe a parabola the line through the middle points of its diagonals will be parallel to the axis of the parabola.

26. If from any point on a focal chord of a parabola two tangents be drawn, these two tangents are equally inclined to the tangents at the extremities of the focal chord.

27. If two tangents to a parabola make equal angles with a fixed straight line, shew that the chord of contact must pass through a fixed point.

28. Two parabolas have a common focus and their axes in opposite directions; prove that the locus of the middle points of chords of either which touch the other is another parabola.

29. Find the locus of the middle point of a chord of a parabola which subtends a right angle at the vertex.

✓ 30. The locus of the middle points of normal chords of the parabola $y^2 - 4ax = 0$ is $\dfrac{y^2}{2a} + \dfrac{4a^3}{y^2} = x - 2a$.

✓ 31. PQ is a chord of a parabola normal at P, AQ is drawn from the vertex A, and through P a line is drawn parallel to AQ meeting the axis in R. Shew that AR is double the focal distance of P.

32. Parallel chords are drawn to a parabola; shew that the locus of the intersection of tangents at the ends of the chords is a straight line, also the locus of the intersection of the normals is a straight line, and the locus of the intersection of these two lines is a parabola.

33. If the normals at the points P, Q, R of a parabola meet in a point; shew that the circle PQR will go through the vertex of the parabola.

34. If the normals at two points of a parabola be inclined to the axis at angles θ, ϕ such that $\tan\theta\tan\phi = 2$, shew that they intersect on the parabola.

35. The locus of a point from which two normals can be drawn making complementary angles with the axis is a parabola.

✓ 36. Two of the normals drawn to a parabola from a point P make equal angles with a given straight line. Prove that the locus of P is a parabola.

✓ 37. The normal at a point P of a parabola meets the axis in G, PG is produced to H, so that $GH = \frac{1}{2}PG$; prove that the other two normals to the parabola, which pass through H, are at right angles to each other.

38. The normals at three points P, Q, R of a parabola meet in the point O. Prove that $SP + SQ + SR + SA = 2OM$, where S is the focus and OM the perpendicular from O on the tangent at the vertex.

39. Any three tangents to a parabola, the tangents of whose inclinations to the axis are in any given harmonical progression, will form a triangle of constant area.

40. Shew that the area of the triangle formed by three normals to a parabola will be

$$\frac{a^2}{2}(m_1 \sim m_2)(m_2 \sim m_3)(m_3 \sim m_1)(m_1 + m_2 + m_3)^2.$$

41. If a tangent to a parabola cut two given parallel straight lines in P, Q, the locus of the point of intersection of the other tangents from P, Q to the curve will be a parabola.

42. If an equilateral triangle circumscribe a parabola, shew that the lines from any vertex to the focus will pass through the point of contact of the opposite side.

43. From any point on $y^2 = a(x + c)$ tangents are drawn to $y^2 = 4ax$; shew that the normals to this parabola at the points of contact intersect on a fixed straight line.

44. If the normals at two points on a parabola intersect on the curve, the line joining the points will pass through a fixed point on the axis.

45. If through a fixed point any chord of a parabola be drawn, and normals be drawn at the ends of the chord, shew that the locus of the point of intersection of the normals is another parabola.

46. If three normals from a point to the parabola $y^2 = 4ax$ cut the axis in points whose distances from the vertex are in arithmetical progression, shew that the point lies on the curve $27ay^2 = 2(x - 2a)^3$.

47. If three of the sides of a quadrilateral inscribed in a parabola be parallel to given straight lines; shew that the fourth side will also be parallel to a fixed straight line.

48. Circles are described on any two focal chords of a parabola as diameters. Prove that their common chord passes through the vertex of the parabola.

49. Two tangents to a given parabola make angles with the axis such that the product of the tangents of their halves is constant; prove that the locus of the point of intersection of the tangents is a confocal parabola.

50. If the circle described on the chord PQ of a parabola as diameter cut the parabola again in the points R, S, then will PQ and RS intercept a constant length on the axis of the parabola.

✓ 51. If the normals at P, Q, R meet in a point O, and PP', QQ', RR' be lines through P, Q, R making with the axis angles equal to those made by PO, QO, RO respectively, then will PP', QQ', RR' pass through another point O', and the line OO' will be perpendicular to the polar of O'.

✓ 52. The normals to a parabola at P, Q, R meet in O; shew that $OP \cdot OQ \cdot OR = a \cdot OL \cdot OM$, where OL and OM are tangents from O to the parabola, and $4a$ is the length of the latus rectum.

53. If from any point in a straight line perpendicular to the axis of a parabola normals be drawn to the curve, prove that the sum of the squares of the sides of the triangle formed by joining the feet of these normals is constant.

54. A triangle ABC is formed by three tangents to a parabola, another triangle DEF is formed by joining the points in which the chords through two points of contact meet the diameter through the third. Shew that A, B, C are the middle point of the sides of DEF.

55. If ABC be a triangle inscribed in a parabola, and $A'B'C'$ be a triangle formed by three tangents parallel to the sides of the triangle ABC, shew that the sides of ABC will be four times the corresponding sides of $A'B'C'$.

56. If four straight lines touch a parabola, shew that the product of the squares of the abscissæ of the point of intersection of two of these tangents and of the point of intersection of the other two is equal to the continued product of the abscissæ of the four points of contact.

57. *TP*, *TQ* are tangents to a parabola, and p_1, p_2, p_3 are the lengths of the perpendiculars from *P*, *T*, *Q* respectively on any other tangent to the curve; shew that $p_1 p_3 = p_2^2$.

58. *OA*, *OB* are tangents to a parabola, and *AP*, *BP* are the corresponding normals; shew that, if *P* lies on a fixed line perpendicular to the axis, *O* describes a parabola; and find the locus of *O*, if *P* lies on a fixed diameter.

✓ 59. *PG* is the normal at *P* to the parabola $y^2 - 4ax = 0$, *G* being on the axis; *GP* is produced outwards to *Q* so that $PQ = GP$; shew that the locus of *Q* is a parabola, and that the locus of the intersection of the tangents at *P* and *Q* is

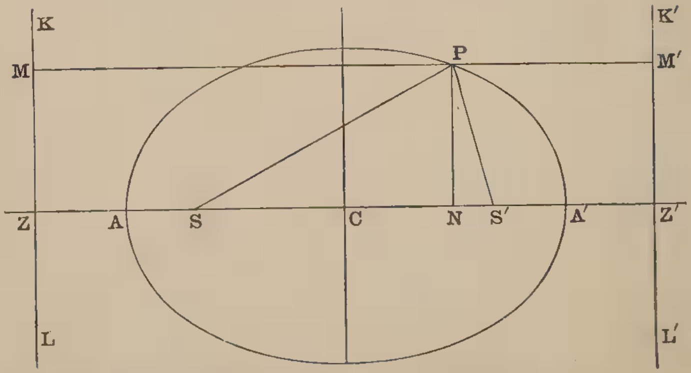
$$y^2 (x + 4a) + 16a^3 = 0.$$

CHAPTER VI.

THE ELLIPSE.

Definition. An *Ellipse* is the locus of a point which moves so that its distance from a fixed point, called the focus, bears a constant ratio, which is less than unity, to its distance from a fixed line, called the directrix.

108. *To find the equation of an ellipse.*



Let S be the focus and KL the directrix.

Draw SZ perpendicular to the directrix.

Divide ZS in A so that $SA : AZ =$ given ratio $= e : 1$ suppose.

There will be a point A' in ZS produced such that

$$SA' : A'Z :: e : 1.$$

Let C be the middle point of AA', and let $AA'=2a$.

Then $SA=e.AZ$, and $SA'=e.A'Z$;

$$\therefore\quad SA+SA'=e(AZ+A'Z);$$

$$\therefore\quad 2AC=2e.ZC;$$

$$\therefore\quad ZC=\frac{a}{e}\text{..........................(i).}$$

Also $$SA'-SA=e(A'Z-AZ);$$

or $$AA'-2AS=e.AA';$$

$$\therefore\quad SC=e.AC=ae\text{ ....(ii).}$$

Now let C be taken as origin, CA' as the axis of x, and a line perpendicular to CA' as the axis of y.

Let P be any point on the curve, and let its co-ordinates be x, y.

Then, in the figure,

$$SP^2=e^2PM^2;$$

$$\therefore\quad SN^2+NP^2=e^2ZN^2.$$

Now $$SN=SC+CN=ae+x;$$

and $$ZN=ZC+CN=\frac{a}{e}+x;$$

$$\therefore\quad (ae+x)^2+y^2=e^2\left(x+\frac{a}{e}\right)^2,$$

or $$y^2+x^2(1-e^2)=a^2(1-e^2),$$

or $$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1\text{.................(iii).}$$

Putting $x=0$, we get $y=\pm a\sqrt{(1-e^2)}$; which gives us the intercepts on the axis of y. If these lengths be called $\pm b$, we have

$$b^2=a^2(1-e^2)\text{ ....(iv),}$$

and the equation (iii) takes the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\text{ ....(v).}$$

The *latus rectum* is the chord through the focus parallel to the directrix. To find its length we must put $x = -ae$ in equation (v).

Then $$y^2 = b^2(1-e^2) = \frac{b^4}{a^2}, \text{ from (iv)},$$

so that the length of the semi-latus rectum is $\dfrac{b^2}{a}$.

109. In equation (v) [Art. 108] the value of y cannot be greater than b, for otherwise x^2 would be negative; and similarly x cannot be greater than a. Hence an ellipse is a curve which is limited in all directions.

If x be numerically less than a, y^2 will be positive; and for any particular value of x there will be two equal and opposite values of y. The axis of x therefore divides the curve into two similar and equal parts.

So also, if y be numerically less than b, x^2 will be positive, and for any particular value of y there will be two values of x which will be equal and opposite. The axis of y therefore divides the curve into two similar and equal parts. From this it follows that if on the axis of x the points S', Z' be taken such that $CS' = SC$, and $CZ' = ZC$, the point S' will also be a focus of the curve, and the line through Z' perpendicular to CZ' will be the corresponding directrix.

If (x', y') be any point on the curve, the co-ordinates x', y' will satisfy the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$; and it is clear that in that case the co-ordinates $-x'$, $-y'$ will also satisfy the equation, so that the point $(-x', -y')$ will also be on the curve. But the points (x', y') and $(-x', -y')$ are on a straight line through the origin and are equidistant from the origin. Hence the origin bisects every chord which passes through it, and is therefore called the *centre* of the curve.

The chord through the foci is called the *major axis*, and the chord through the centre perpendicular to this the *minor axis*.

110. *To find the focal distances of any point on an ellipse.*

In the figure to Art. 108, since $SP = ePM$, we have

$$SP = eZN = e(ZC + CN) = e\left(\frac{a}{e} + x\right) = a + ex;$$

also $$S'P = e\,.\,NZ' = e(CZ' - CN) = a - ex;$$

$$\therefore\ SP + S'P = 2a.$$

An ellipse is sometimes *defined* as the locus of a point which moves so that the sum of its distances from two fixed points is constant.

To find the equation of the curve from this definition.

Let the constant sum be $2a$, and the distance between the two fixed points be $2ae$.

Take the middle point of the line joining the fixed points for origin, and this line and a line perpendicular to it for axes, then we have from the given condition

$$\sqrt{(x - ae)^2 + y^2} + \sqrt{(x + ae)^2 + y^2} = 2a$$

which, when rationalized, becomes

$$y^2 + x^2(1 - e^2) = a^2(1 - e^2),$$

which is the equation previously obtained.

111. The polar equation of the ellipse referred to the centre as pole will be found by writing $r\cos\theta$ for x, and $r\sin\theta$ for y in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The equation will therefore be

$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1,$$

or $$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

The equation (i) can be written in the form

$$\frac{1}{r^2} = \frac{1}{a^2} + \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\sin^2\theta \quad\ldots\ldots\ldots\ldots\text{(ii)}.$$

Since $\frac{1}{b^2} - \frac{1}{a^2}$ is positive, we see from (ii) that the least value of $\frac{1}{r^2}$ is $\frac{1}{a^2}$, and that $\frac{1}{r^2}$ increases as θ increases from 0 to $\frac{\pi}{2}$, the greatest value of $\frac{1}{r^2}$ being $\frac{1}{b^2}$. Hence the radius vector diminishes from a to b as θ increases from 0 to $\frac{\pi}{2}$.

112. We have found that for all points on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

We can shew in a manner similar to that adopted in Art. 92 that, if x, y be the co-ordinates of any point *within* the curve, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ will be *negative*, and that $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ will be *positive* if x, y be the co-ordinates of any point *outside* the curve.

113. *To find the points of intersection of a given straight line and an ellipse, and to find the condition that a given straight line may touch the ellipse.*

NOTE. We shall henceforth always take $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as the equation of the ellipse, unless it is otherwise expressed.

Let the equation of the straight line be

$$y = mx + c \quad\text{..........................(i).}$$

At points which are common to the straight line and the ellipse both these relations are satisfied. Hence at the common points we have

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1,$$

or $$x^2 (b^2 + a^2m^2) + 2mca^2x + a^2 (c^2 - b^2) = 0 \quad\text{......(ii).}$$

This is a quadratic equation, and every quadratic equation has *two* roots, real, coincident, or imaginary.

Hence there are two values of x, and the two corresponding values of y are given by equation (i).

The roots of the equation (ii) will be equal to one another, if

$$a^2(c^2-b^2)(b^2+a^2m^2)=m^2c^2a^4,$$

that is, if

$$c^2=a^2m^2+b^2.$$

If the two values of x are equal to one another the two values of y must also be equal to one another from (i).

Therefore the two points in which the ellipse is cut by the line will be *coincident* if $c=\sqrt{(a^2m^2+b^2)}$.

Hence the line whose equation is

$$y=mx+\sqrt{(a^2m^2+b^2)}\ \ldots\ldots\ldots\ldots\ldots(\text{iii})$$

will touch the ellipse for all values of m.

Since either sign may be given to the radical in (iii), it follows that there are *two* tangents to the ellipse for every value of m, that is, there are two tangents parallel to any given straight line. These two parallel tangents are equidistant from the centre of the ellipse.

114. *To find the equation of the chord joining two points on the ellipse, and to find the equation of the tangent at any point.*

Let x', y' and x'', y'' be the co-ordinates of two points on the ellipse.

The equation of the secant through the points (x', y') and (x'', y'') is

$$\frac{x-x'}{x''-x'}=\frac{y-y'}{y''-y'},\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{i}).$$

But, since the two points are on the ellipse, we have

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1,\ \text{ and }\ \frac{x''^2}{a^2}+\frac{y''^2}{b^2}=1;$$

$$\therefore\ \frac{x''^2-x'^2}{a^2}=-\frac{y''^2-y'^2}{b^2}\ \ldots\ldots\ldots\ldots(\text{ii}).$$

Multiply the corresponding sides of the equations (i) and (ii), and we have

$$\frac{(x-x')(x''+x')}{a^2}=-\frac{(y-y')(y''+y')}{b^2},$$

or $$\frac{x(x'+x'')}{a^2}+\frac{y(y'+y'')}{b^2}=\frac{x'(x'+x'')}{a^2}+\frac{y'(y'+y'')}{b^2},$$

or, since, $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$,

$$\frac{x(x'+x'')}{a^2}+\frac{y(y'+y'')}{b^2}=1+\frac{x'x''}{a^2}+\frac{y'y''}{b^2}\ \ldots\text{(iii)},$$

which is the equation of the chord joining the two given points.

In order to find the equation of the tangent at (x', y') we must put $x''=x'$, and $y''=y'$ in equation (iii), and we obtain

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1\ \ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

Cor. 1. The co-ordinates of the extremities of the major axis are a, 0 and $-a$, 0 respectively, and, from (iv), the tangents at these points are $x=a$ and $x=-a$.

Hence the tangents at the extremities of the major axis are parallel to the minor axis.

Similarly the tangents at the extremities of the minor axis are parallel to the major axis.

Cor. 2. The tangent at the point (x', y') is parallel to the tangent at the point $(-x', -y')$, and these two points are on a straight line through the centre of the curve.

Hence the tangents at the extremities of any chord through the centre of an ellipse are parallel to one another.

115. *To find the condition that the line $lx+my-n=0$ may touch the ellipse.*

The equation of the lines joining the origin to the points where the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

is cut by the straight line

$$lx+my=n\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

is [Art. 38] $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\left(\frac{lx+my}{n}\right)^2=0\ \ldots\ldots\ldots\text{(iii)}.$$

If the line (ii) cut the ellipse in *coincident* points, the equation (iii) will represent *coincident* straight lines. Therefore the left-hand member of (iii) must be a perfect square: the condition for this is

$$\left(\frac{1}{a^2}-\frac{l^2}{n^2}\right)\left(\frac{1}{b^2}-\frac{m^2}{n^2}\right)=\frac{l^2m^2}{n^4};$$

whence

$$a^2l^2+b^2m^2=n^2 \text{ ....(iv).}$$

Cor. The line $x\cos\alpha+y\sin\alpha-p=0$ will touch the ellipse, if

$$p^2=a^2\cos^2\alpha+b^2\sin^2\alpha \text{ ....(v).}$$

116. *To find the equation of the normal at any point of an ellipse.*

The equation of the tangent at any point (x', y') of the ellipse is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1.$$

The *normal* is the line through (x', y') perpendicular to the tangent; its equation is therefore [Art. 30]

$$\frac{x-x'}{\dfrac{x'}{a^2}}=\frac{y-y'}{\dfrac{y'}{b^2}}.$$

EXAMPLES.

1. Find the eccentricities, and the co-ordinates of the foci of the following ellipses:

(i) $2x^2+3y^2-1=0$, (ii) $8(x-1)^2+6(y+1)^2-1=0$.

Ans. (i) $\frac{1}{\sqrt{3}}$, $\left(\pm\frac{1}{\sqrt{6}}, 0\right)$; (ii) $\frac{1}{2}$, $(1, -1\pm\frac{1}{12}\sqrt{6})$.

2. Find the lengths of the latera recta of the ellipses in question 1.

Ans. $\frac{2}{3}\sqrt{2}$ and $\frac{1}{4}\sqrt{6}$.

3. Shew that the line $y=x+\sqrt{\frac{5}{6}}$ touches the ellipse $2x^2+3y^2=1$.

4. Shew that the line $3y=x-3$ cuts the curve $4x^2-3y^2-2x=0$ in two points equidistant from the axis of y.

5. Is the point (2, 1) within or without the ellipse $2x^2+3y^2-12=0$?

6. Find the equations of the tangents to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which make an angle of 60^0 with the axis of x.

7. Find (i) the equations of the tangents and (ii) the equations of the normals at the ends of the latera recta of $2x^2+3y^2=6$.

The four points are $(\pm 1, \ \pm\frac{2}{3}\sqrt{3})$.

8. Find the equations of the tangents to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which make equal intercepts on the axes. *Ans.* $x\pm y\pm\sqrt{a^2+b^2}=0$.

9. Shew that the equation $4x^2+2y^2=6x$ represents an ellipse whose eccentricity is $\frac{1}{\sqrt{2}}$, and shew that the origin is at an extremity of the minor axis.

10. Find the equation of the ellipse which has the point $(-1, 1)$ for focus, the line $4x-3y=0$ for directrix, and whose eccentricity is $\frac{5}{6}$.
Ans. $20x^2+24xy+27y^2+72(x-y+1)=0$.

11. If the normal at the end of a latus rectum of an ellipse pass through one extremity of the minor axis, shew that the eccentricity of the curve is given by the equation $e^4+e^2-1=0$.

12. If any ordinate MP be produced to meet the tangent at the end of the latus rectum through the focus S in Q, shew that the ordinate of Q is equal to the distance SP.

13. A straight line AB of given length has its extremities on two fixed straight lines OA, OB which are at right angles; shew that the locus of any point C on the line is an ellipse whose semi-axes are equal to CA and CB respectively.

117. *Two tangents can be drawn to an ellipse from any point, which will be real, coincident, or imaginary, according as the point is outside, upon, or within the curve.*

The line whose equation is

$$y=mx+\sqrt{(a^2m^2+b^2)}\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

will touch the ellipse, whatever the value of m may be [Art. 113].

The line (i) will pass through the particular point (x', y'), if

$$y' = mx' + \sqrt{(a^2m^2 + b^2)},$$

that is, if

$$(y' - mx')^2 - a^2m^2 - b^2 = 0,$$

or $$m^2(x'^2 - a^2) - 2mx'y' + y'^2 - b^2 = 0 \text{ .....(ii)}.$$

Equation (ii) is a quadratic equation which gives the directions of those tangents to the ellipse which pass through the point (x', y'). Since a quadratic equation has *two* roots, *two* tangents will pass through any point (x', y').

The roots of (ii) are real, coincident, or imaginary, according as

$$(x'^2 - a^2)(y'^2 - b^2) - x'^2y'^2$$

is negative, zero, or positive; or according as $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1$ is positive, zero, or negative. That is, [Art. 112] according as (x', y') is outside the ellipse, upon the ellipse, or within it.

118. *To find the equation of the line through the points of contact of the two tangents which can be drawn to an ellipse from any point.*

Let x', y' be the co-ordinates of the point from which the tangents are drawn.

Let the co-ordinates of the points of contact of the tangents be h, k and h', k' respectively.

The equations of the tangents at (h, k) and (h', k') are

$$\frac{xh}{a^2} + \frac{yk}{b^2} = 1,$$

and $$\frac{xh'}{a^2} + \frac{yk'}{b^2} = 1.$$

We know that (x', y') is on both these lines;

$$\therefore \quad \frac{x'h}{a^2} + \frac{y'k}{b^2} = 1 \text{ ....(i)},$$

and $$\frac{x'h'}{a^2} + \frac{y'k'}{b^2} = 1 \text{ ....(ii)}.$$

But (i) and (ii) shew that the points (h, k) and (h', k') are both on the straight line whose equation is

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} = 1 \qquad \text{(iii)}.$$

Hence (iii) is the required equation of the line through the points of contact of the tangents from (x', y').

The line joining the points of contact of the two tangents from any point P to an ellipse is called the *polar* of P with respect to the ellipse. [See Art. 76.]

119. *If the polar of a point P with respect to an ellipse pass through the point Q, then will the polar of Q pass through P.*

This may be proved exactly as in Art. 78.

120. *To find the locus of the point of intersection of two tangents to an ellipse which are at right angles to one another.*

The line whose equation is

$$y = mx + \sqrt{a^2m^2 + b^2} \qquad \text{(i)}$$

will touch the ellipse, whatever the value of m may be.

If we suppose x and y to be known, the equation gives us the *directions* of the tangents which pass through the point (x, y).

The equation, when rationalized, becomes

$$m^2(x^2 - a^2) - 2mxy + y^2 - b^2 = 0 \qquad \text{(ii)}.$$

Let m_1 and m_2 be the roots of (ii); then, if the tangents be at right angles, $m_1m_2 = -1$;

$$\therefore \quad \frac{y^2 - b^2}{x^2 - a^2} = -1,$$

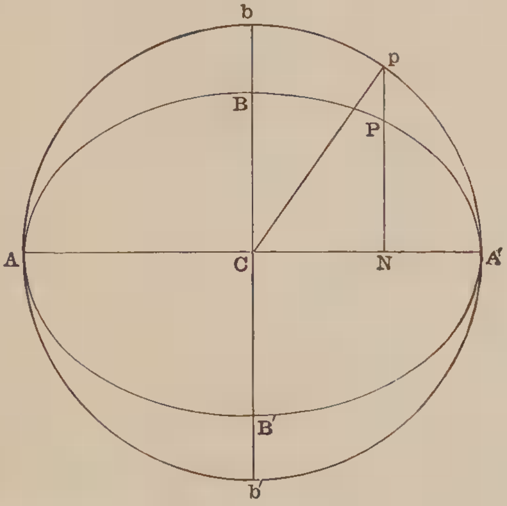
or

$$x^2 + y^2 = a^2 + b^2 \qquad \text{(iii)}.$$

The required locus is therefore a circle.

This circle is called the *director circle* of the ellipse.

121. The circle described on the major axis of an ellipse as diameter is called the *auxiliary circle.*



If the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{......................(i)},$$

the equation of the auxiliary circle will be

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1 \text{......................(ii)}.$$

If therefore any ordinate NP of the ellipse be produced to meet the auxiliary circle in p, we have from (i) and (ii)

$$\frac{NP^2}{b^2}=1-\frac{CN^2}{a^2}=\frac{Np^2}{a^2};$$

$$\therefore \quad \frac{NP}{Np}=\frac{b}{a}.$$

Hence the ordinates of the ellipse and of the circle are in a constant ratio to one another.

The angle $A'Cp$ is called the *eccentric angle* of the point P. The point p on the auxiliary circle is said to *correspond* to the point P on the ellipse.

If the angle $A'Cp$ be ϕ, the co-ordinates of p will be $a\cos\phi$, $a\sin\phi$; and those of P will be $a\cos\phi$, $b\sin\phi$.

122. *To find the equation of the line joining two points whose eccentric angles are given.*

Let ϕ, ϕ' be the eccentric angles of the two points; then the co-ordinates are $a\cos\phi$, $b\sin\phi$, and $a\cos\phi'$, $b\sin\phi'$ respectively.

Hence the equation of the line joining them is

$$\frac{x - a\cos\phi}{a\cos\phi - a\cos\phi'} = \frac{y - b\sin\phi}{b\sin\phi - b\sin\phi'},$$

or

$$\frac{\dfrac{x}{a} - \cos\phi}{-\sin\frac{1}{2}(\phi + \phi')} = \frac{\dfrac{y}{b} - \sin\phi}{\cos\frac{1}{2}(\phi + \phi')};$$

$$\therefore \frac{x}{a}\cos\tfrac{1}{2}(\phi + \phi') + \frac{y}{b}\sin\tfrac{1}{2}(\phi + \phi') = \cos\tfrac{1}{2}(\phi - \phi') \ldots \text{(i)},$$

which is the required equation.

To find the tangent at the point ϕ, we have to put $\phi' = \phi$ in equation (i), and we obtain

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1 \ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

123. From equation (i) of the preceding article we see that if the sum of the eccentric angles of two points on an ellipse is constant and equal to 2α, the chord joining those points is always parallel to the line

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1;$$

that is, the chord is always parallel to the tangent at the point whose eccentric angle is α.

Conversely, if we have a system of parallel chords of an ellipse the sum of the eccentric angles of the extremities of any chord is constant.

124. *To find the equation of the normal at any point of an ellipse in terms of the eccentric angle of the point.*

Let ϕ be the eccentric angle of a point P on the ellipse; the equation of the tangent at P is [Art. 122]

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1.$$

The equation of the line through $(a\cos\phi,\ b\sin\phi)$ perpendicular to the tangent is [Art. 30]

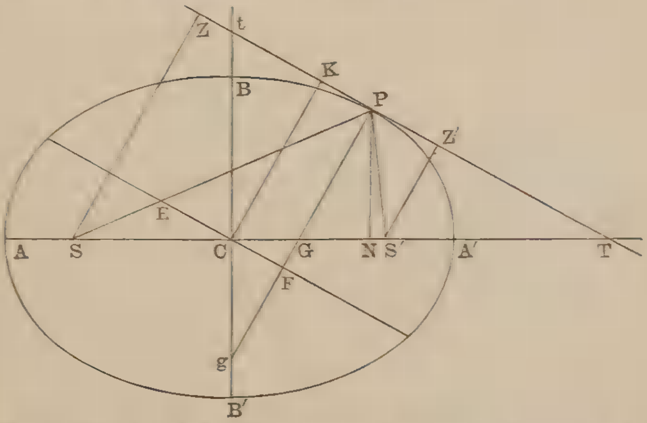
$$(x - a\cos\phi)\frac{a}{\cos\phi} - (y - b\sin\phi)\frac{b}{\sin\phi} = 0,$$

or

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2.$$

125. We will now prove some geometrical properties of an ellipse.

Let the tangent at P meet the axes of x and y in T, t respectively, and let the normal meet the axes in G, g. Draw SZ, $S'Z'$, CK perpendicular to the tangent at P; draw also CE parallel to the tangent at P, meeting the normal in F, and the focal distance SP in E.



Then if x', y' be the co-ordinates of the point P, the equation of the tangent at P will be

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Where this cuts the axis of x, $y=0$, and at that point we have from (i),

$$\frac{xx'}{a^2}=1\,;$$

$$\therefore\quad \frac{CN\,.\,CT}{CA'^2}=1,\ \text{or}\ CN\,.\,CT=CA'^2 \ldots\ldots\ldots(\alpha).$$

Similarly $$NP\,.\,Ct=CB^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta).$$

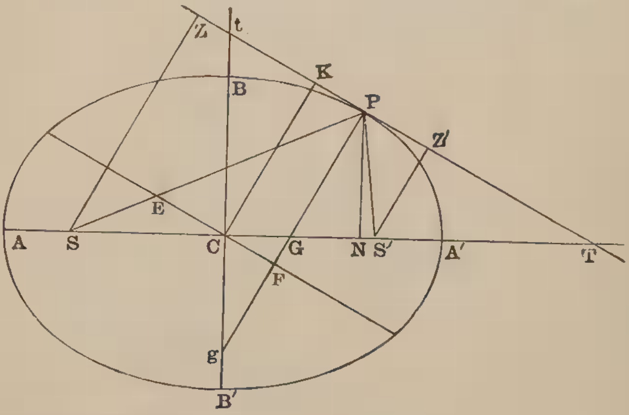
The equation of the normal at P is

$$\frac{x-x'}{\frac{x'}{a^2}}=\frac{y-y'}{\frac{y'}{b^2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii}).$$

Where the normal cuts the axis of x, we have $y=0$, and therefore from (ii),

$$x-x'=-\frac{b^2}{a^2}x',\ \text{or}\ x=x'\left(1-\frac{b^2}{a^2}\right)=e^2x'\,;$$

$$\therefore\quad CG=e^2\,.\,CN \ldots\ldots\ldots\ldots\ldots\ldots(\gamma).$$



Also, since

$$SG=SC+CG=ae+e^2x',\ \text{and}\ GS'=ae-e^2x',$$

we have

$$\frac{SG}{GS'}=\frac{ae+e^2x'}{ae-e^2x'}=\frac{a+ex'}{a-ex'}=\frac{SP}{S'P}\,;$$

therefore PG bisects the angle SPS' ....(δ).

Again, since $PG^2 = GN^2 + NP^2 = (CN - CG)^2 + NP^2$,

we have $$PG^2 = y'^2 + x'^2(1 - e^2)^2,$$

or $$PG = b^2 \sqrt{\left(\frac{y'^2}{b^4} + \frac{x'^2}{a^4}\right)}.$$

And $PF = KC = \dfrac{1}{\sqrt{\left(\dfrac{x'^2}{a^4} + \dfrac{y'^2}{b^4}\right)}}$ [Art. 31];

$$\therefore \quad PF \,.\, PG = b^2 \ldots\ldots\ldots\ldots\ldots\ldots(\epsilon).$$

The line whose equation is

$$y = mx + \sqrt{(a^2m^2 + b^2)} \ldots\ldots\ldots\ldots\ldots(\text{iii})$$

will touch the ellipse whatever the value of m may be.

Hence, if SZ, $S'Z'$ be the perpendiculars from the foci on the line (iii), then [Art. 31]

$$SZ = \frac{-mae + \sqrt{(a^2m^2 + b^2)}}{\sqrt{(1 + m^2)}}, \text{ and } S'Z' = \frac{mae + \sqrt{(a^2m^2 + b^2)}}{\sqrt{(1 + m^2)}};$$

$$\therefore SZ \,.\, S'Z' = \frac{a^2m^2 + b^2 - m^2a^2e^2}{1 + m^2} = b^2 \ldots\ldots\ldots(\zeta).$$

Again, the equation of the line through S perpendicular to (iii) is

$$my + x + ae = 0 \ldots\ldots\ldots\ldots\ldots\ldots(\text{iv}).$$

To find the locus of Z the point of intersection of (iii) and (iv), we must eliminate m from the two equations. The equations may be written in the form

$$y - mx = \sqrt{(a^2m^2 + b^2)}, \text{ and } my + x = -ae.$$

Square both sides of these equations and add, we thus obtain

$$(x^2 + y^2)(1 + m^2) = a^2m^2 + b^2 + a^2e^2 = a^2(1 + m^2);$$

therefore the locus of Z is the auxiliary circle whose equation is

$$x^2 + y^2 = a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\eta).$$

We should have arrived at the same result if we had supposed the perpendicular to have been drawn from S'.

126. Let P be any point, and let QQ' be the polar of P. Let QQ' meet the axes in T, t. Draw SZ, $S'Z'$, CK and PO perpendicular to QQ'; and let PO meet the axes in G, g. Then, if x', y' be the co-ordinates of P, the equation of QQ' will be [Art. 118]

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1 \qquad\qquad \text{(i)}.$$

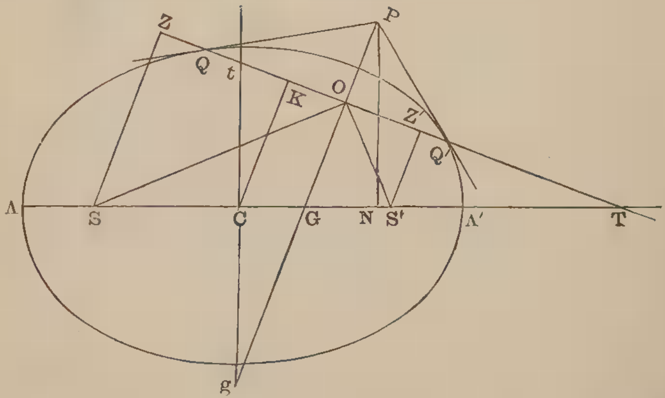
The equation of POG will therefore be [Art. 30]

$$\frac{x-x'}{\dfrac{x'}{a^2}}=\frac{y-y'}{\dfrac{y'}{b^2}} \qquad\qquad \text{(ii)}.$$

From (i) and (ii) we can prove, exactly as in the preceding Article,

(α) $CN\,.\,CT=CA^2$, (β) $NP\,.\,Ct=CB^2$,

(γ) $CG=e^2\,CN$, and (δ) $KC\,.\,PG=b^2$.



EXAMPLES.

1. Shew that the focus of an ellipse is the pole of the corresponding directrix.

2. Shew that the equation of the locus of the foot of the perpendicular from the centre of an ellipse on a tangent is $r^2=a^2\cos^2\theta+b^2\sin^2\theta$.

3. Shew that the sum of the reciprocals of the squares of any two diameters of an ellipse which are at right angles to one another is constant. [See Art. 111.]

4. If an equilateral triangle be inscribed in an ellipse the sum of the squares of the reciprocals of the diameters parallel to the sides will be constant.

5. An ellipse slides between two straight lines at right angles to one another; shew that the locus of its centre is a circle. [See Art. 120.]

6. If the points S', H' be taken on the minor axis of an ellipse such that $S'C = CH' = CS$, where C is the centre and S is a focus; shew that the sum of the squares of the perpendiculars from S' and H' on any tangent to the ellipse is constant.

7. Shew that the locus of the point of intersection of tangents to an ellipse at two points whose eccentric angles differ by a constant is an ellipse.

[If the tangents at $\phi + \alpha$ and $\phi - \alpha$ meet at (x', y'); then $\frac{x'}{a} = \cos\phi \sec\alpha$, $\frac{y'}{b} = \sin\phi \sec\alpha$. Eliminate ϕ for the locus.]

8. The polar of a point P cuts the minor axis in t, and the perpendicular from P to its polar cuts the polar in the point O and the minor axis in g; shew that the circle through the points t, O, g will pass through the foci. [Prove that $tC \cdot Cg = SC \cdot CS'$.]

9. Prove that the line $lx + my + n = 0$ is a normal to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ if } \frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}.$$

[Compare with $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$; we have $\frac{l\cos\theta}{a} = -\frac{m\sin\theta}{b} = \frac{n}{a^2 - b^2}$: then eliminate θ.]

10. The perpendicular from the focus of an ellipse whose centre is C on the tangent at any point P will meet the line CP on the directrix.

11. If Q be the point on the auxiliary circle corresponding to the point P on an ellipse, shew that the normals at P and Q meet on a fixed circle.

12. If Q be the point on the auxiliary circle corresponding to the point P on an ellipse, shew that the perpendicular distances of the foci S, H from the tangent at Q are equal to SP and HP respectively.

13. Shew that the area of a triangle inscribed in an ellipse is

$$\tfrac{1}{2} ab \{\sin(\beta - \gamma) + \sin(\gamma - \alpha) + \sin(\alpha - \beta)\},$$

where α, β, γ are the eccentric angles of the angular points.

127. *To find the locus of the middle points of a system of parallel chords of an ellipse.*

The equation of the line through the two points (x', y') and (x'', y'') on the ellipse is [Art. 114]

$$\frac{x(x'+x'')}{a^2}+\frac{y(y'+y'')}{b^2}=1+\frac{x'x''}{a^2}+\frac{y'y''}{b^2}\;\ldots\ldots\ldots\text{(i)}.$$

If the line (i) is parallel to the line $y=mx$, then

$$m=-\frac{b^2}{a^2}\frac{x'+x''}{y'+y''}\;\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Now, if x, y be the co-ordinates of the middle point of the chord joining (x', y') and (x'', y''), then $2x=x'+x''$, and $2y=y'+y''$;

therefore, from (ii), we have

$$m=-\frac{b^2x}{a^2y}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Hence the locus of the middle points of all chords which are parallel to the line $y=mx$ is the straight line whose equation is

$$y=-\frac{b^2}{a^2m}x\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

From (iv) we see that all *diameters* of an ellipse [Art. 102, *Def.*] pass through the centre.

Writing (iv) in the form $y=m'x$, we see that

$$mm'=-\frac{b^2}{a^2}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}.$$

It is clear from the symmetry of the relation (v) that all chords parallel to $y=m'x$ are bisected by $y=mx$.

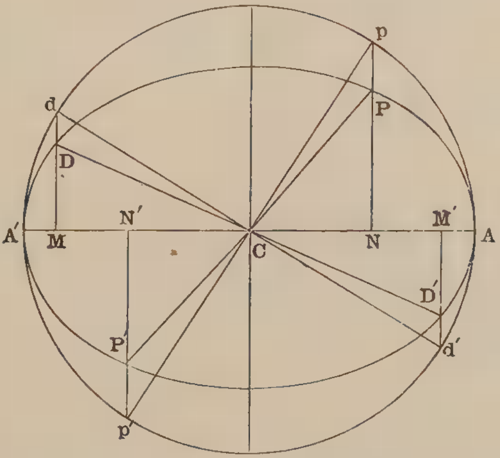
Hence, *if one diameter of an ellipse bisect chords parallel to a second, the second diameter will bisect all chords parallel to the first.*

Def. Two diameters are said to be *conjugate* when each bisects chords parallel to the other.

128. *The tangent at an extremity of any diameter is parallel to the chords bisected by that diameter.*

All the middle points of a system of parallel chords of an ellipse are on a diameter. Hence, by considering the parallel tangents, that is the parallel chords which cut the curve in coincident points, we see that the diameter of a system of parallel chords passes through the points of contact of the tangents which are parallel to the chords.

129. Let P, D be extremities of a pair of conjugate diameters; let the co-ordinates of P be x', y', and the co-ordinates of D be x'', y''. The equations of CP and CD are
$$\frac{y}{y'} = \frac{x}{x'} \text{ and } \frac{y}{y''} = \frac{x}{x''};$$
hence from (v) Art. 127 we have $\frac{y'y''}{x'x''} = -\frac{b^2}{a^2}$;
$$\therefore \quad \frac{x'x''}{a^2} + \frac{y'y''}{b^2} = 0 \text{......................(i).}$$



If ϕ, ϕ' be the eccentric angles of P and D respectively, then $x' = a \cos \phi$, $y' = b \sin \phi$, $x'' = a \cos \phi'$, $y'' = b \sin \phi'$. Substituting these values in (i) we have
$$\cos \phi \cos \phi' + \sin \phi \sin \phi' = 0,$$
or
$$\phi \sim \phi' = \frac{\pi}{2}.$$

Hence the difference of the eccentric angles of two points which are at extremities of two conjugate diameters of an ellipse is a right angle.

If pCp', dCd' be the diameters of the auxiliary circle corresponding to the diameters PCP', DCD' of the ellipse, then pCp', dCd' will be at right angles to one another. Hence the co-ordinates of D and of D' can be at once expressed in terms of those of P or of P'.

130. *To shew that the sum of the squares of two conjugate semi-diameters is constant.*

Let P, D be extremities of two conjugate diameters of the ellipse.

Let the eccentric angle of P be ϕ, then the eccentric angle of D will be $\phi \pm \frac{\pi}{2}$ [Art. 129].

The co-ordinates of P will be $a\cos\phi$, $b\sin\phi$, and those of D will be $a\cos\left(\phi \pm \frac{\pi}{2}\right)$, $b\sin\left(\phi \pm \frac{\pi}{2}\right)$.

$$\therefore \quad CP^2 = a^2\cos^2\phi + b^2\sin^2\phi,$$

and
$$CD^2 = a^2\cos^2\left(\phi \pm \frac{\pi}{2}\right) + b^2\sin^2\left(\phi \pm \frac{\pi}{2}\right);$$

$$\therefore \quad CP^2 + CD^2 = a^2 + b^2.$$

131. *The area of the parallelogram which touches an ellipse at the ends of conjugate diameters is constant.*

Let PCP', DCD' be the conjugate diameters. The area of the parallelogram which touches the ellipse at P, P', D, D' is $4CP \,.\, CD \sin PCD$, or $4CD \,.\, CF$ where CF is the perpendicular from C on the tangent at P.

Now if the eccentric angle of P be ϕ, the eccentric angle of D will be $\phi \pm \frac{\pi}{2}$.

$$\therefore \quad CD^2 = a^2\cos^2\left(\phi \pm \frac{\pi}{2}\right) + b^2\sin^2\left(\phi \pm \frac{\pi}{2}\right),$$

or
$$CD^2 = a^2\sin^2\phi + b^2\cos^2\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

The equation of the tangent at P will be [Art. 122]

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1.$$

$$\therefore\quad CF^2=\frac{1}{\dfrac{\cos^2\phi}{a^2}+\dfrac{\sin^2\phi}{b^2}},\quad \text{[Art. 31]},$$

or
$$CF^2=\frac{a^2b^2}{a^2\sin^2\phi+b^2\cos^2\phi}\ \ldots\ldots\ldots\ldots\text{(ii)}.$$

From (i) and (ii) we see that the area of the parallelogram is equal to $4ab$.

132. If r, r' be the lengths of a pair of conjugate semi-diameters, and θ be the angle between them, then

$$rr'\sin\theta=ab\ \text{[Art. 131]}.$$

Hence $\sin\theta$ is least when rr' is greatest.

Now the sum of the squares of two conjugate diameters is constant; hence the product will be greatest when the diameters are equal to one another.

Hence the acute angle between two conjugate diameters of an ellipse is least when the conjugate diameters are equal to one another.

133. Let the eccentric angles of the extremities P, D of two conjugate diameters be ϕ, $\phi\pm\frac{\pi}{2}$ respectively; then

$$CP^2=a^2\cos^2\phi+b^2\sin^2\phi,$$

and
$$CD^2=a^2\sin^2\phi+b^2\cos^2\phi;$$

$$\therefore\ CP^2-CD^2=(a^2-b^2)\cos 2\phi.$$

Hence
$$CP=CD \text{ when } \phi \text{ is } \frac{\pi}{4} \text{ or } \frac{3\pi}{4}.$$

The equations of the *equal* conjugate diameters are therefore

$$\frac{x}{a}=\pm\frac{y}{b}.$$

Hence the equi-conjugate diameters of an ellipse are coincident in direction with the diagonals of the rectangle formed by the tangents at the ends of its axes.

134. *Def.* The two straight lines drawn from any point on an ellipse to the extremities of any diameter are called *supplemental* chords.

Any two supplemental chords of an ellipse are parallel to a pair of conjugate diameters.

Let the chords be formed by joining the point Q to the extremities P, P' of the diameter PCP'. Let V be the middle point of QP, and V' the middle point of QP'. Then CV' and CV are conjugate, for each bisects a chord parallel to the other; and CV', CV are parallel respectively to QP and QP'.

Hence QP and QP' are parallel to a pair of conjugate diameters.

135. We can find the position of a pair of conjugate diameters which make a given angle with one another.

For, draw any diameter PCP' and on PP' describe a segment of a circle containing the given angle. If this circle cut the ellipse in a point Q, the angle PQP' is equal to the given angle, and QP, QP' are parallel to conjugate diameters [Art. 134].

The circle and ellipse will not however intersect in any real points besides the points P, P' if the given angle be less than that between the equi-conjugate diameters of the ellipse [Art. 132].

136. *To find the equation of an ellipse referred to any pair of conjugate diameters as axes.*

Let the equation of the ellipse referred to its major and minor axes be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ ....(i).}$$

Since the origin is unaltered we substitute for x, y expressions of the form $lx+my$, $l'x+m'y$ in order to obtain the transformed equation [Art. 51].

The equation of the ellipse will therefore be of the form $$Ax^2 + 2Hxy + By^2 = 1 \text{ ....(ii).}$$

By supposition the axis of x bisects all chords parallel to the axis of y. Therefore for *any* particular value of x the two values of y found from (ii) must be equal and of opposite sign. Hence $H = 0$; the equation will therefore be of the form $$Ax^2 + By^2 = 1 \text{ ....(iii).}$$

To obtain the lengths (a', b') of the intercepts on the axes of x, y, we must put $y = 0$ and $x = 0$ in (iii); we thus obtain $$Aa'^2 = 1 = Bb'^2.$$

Hence the equation of an ellipse referred to conjugate diameters is

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1,$$

where a', b' are the lengths of the semi-diameters.

137. By the preceding Article we see that when an ellipse is referred to any pair of conjugate diameters as axes of co-ordinates, its equation is of the same form as when its major and minor axes are the axes of co-ordinates.

It will be seen that Articles 113, 114, 115, 118 and 127, hold good when the axes of co-ordinates are *any* pair of conjugate diameters.

138. We shall conclude this chapter by the solution of some examples.

(1) *To find when the area of a triangle inscribed in an ellipse is greatest.*

Let the eccentric angles of P, Q, R, the angular points of the triangle, be ϕ_1, ϕ_2, ϕ_3; let p, q, r be the three corresponding points on the auxiliary circle.

The areas of the triangles PQR, and pqr are [Art. 6]

$$\frac{1}{2}\begin{vmatrix} a\cos\phi_1, & b\sin\phi_1, & 1 \\ a\cos\phi_2, & b\sin\phi_2, & 1 \\ a\cos\phi_3, & b\sin\phi_3, & 1 \end{vmatrix}, \text{ and } \frac{1}{2}\begin{vmatrix} a\cos\phi_1, & a\sin\phi_1, & 1 \\ a\cos\phi_2, & a\sin\phi_2, & 1 \\ a\cos\phi_3, & a\sin\phi_3, & 1 \end{vmatrix};$$

$$\therefore \triangle PQR : \triangle pqr :: b : a.$$

Hence the triangles PQR and pqr are to one another in the constant ratio $b : a$. Therefore PQR is greatest when pqr is greatest.

Now $\triangle pqr$ is greatest when it is an *equilateral* triangle; and in that case $\phi_1 \sim \phi_2 = \phi_2 \sim \phi_3 = \phi_3 \sim \phi_1 = \frac{2\pi}{3}$.

Hence when a triangle inscribed in an ellipse is a maximum, the eccentric angles of its angular points are α, $\alpha + \frac{2\pi}{3}$, $\alpha + \frac{4\pi}{3}$.

(2) *If any pair of conjugate diameters of an ellipse cut the tangent at a point P in T, T'; shew that $TP \cdot PT' = CD^2$, where CD is the diameter conjugate to CP.*

Take CP, CD for axes of x and y, then the equation of the ellipse will be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The equation of the tangent at $P\,(a, 0)$ will be $x = a$.

If $y = mx$, $y = m'x$ be the equations of any pair of conjugate diameters, then

$$mm' = -\frac{b^2}{a^2} \text{ [Art. 127]} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

But
$$PT = ma, \text{ and } PT' = m'a;$$
$$\therefore PT \cdot PT' = mm'a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$
$$\therefore TP \cdot PT' = b^2, \text{ from (i)}.$$

(3) *The line joining the extremities of any two diameters of an ellipse which are at right angles to one another will always touch a fixed circle.*

Let CP, CQ be two diameters which are at right angles to one another, and let the equation of the line PQ be

$$x \cos\alpha + y \sin\alpha = p.$$

The equation of the lines CP, CQ will be [Art. 38]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

But, since the lines CP, CQ are at right angles to one another, the sum of the coefficients of x^2 and y^2 in (i) is zero [Art. 36];

$$\therefore \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2},$$

which shews that the perpendicular distance of the line PQ from the centre is constant.

Hence the line PQ always touches a fixed circle.

(4) *To find the locus of the poles of normal chords of an ellipse.*

The equation of the normal at any point θ is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \qquad \text{(i)}.$$

The equation of the polar of any point (x', y') is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1 \qquad \text{(ii)}.$$

The equations (i) and (ii) will represent the same straight line, if

$$(a^2 - b^2)\frac{x'}{a^2} = \frac{a}{\cos\theta}, \text{ and } (a^2 - b^2)\frac{y'}{b^2} = -\frac{b}{\sin\theta};$$

or
$$(a^2 - b^2)\cos\theta = \frac{a^3}{x'}, \text{ and } (a^2 - b^2)\sin\theta = -\frac{b^3}{y'};$$

therefore, by squaring and adding the two last equations, we have

$$(a^2 - b^2)^2 = \frac{a^6}{x'^2} + \frac{b^6}{y'^2}.$$

Hence the equation of the locus is

$$x^2y^2(a^2 - b^2)^2 = a^6y^2 + b^6x^2.$$

(5) *If a quadrilateral circumscribe an ellipse, the line through the middle points of its diagonals will pass through the centre of the ellipse.*

Let the eccentric angles of the four points of contact of the tangents be α, β, γ, δ.

The equations of the tangents at the points α, β are

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1, \text{ and } \frac{x}{a}\cos\beta + \frac{y}{b}\sin\beta = 1.$$

These meet in the point

$$\left(a\frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}, b\frac{\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}\right).$$

The tangents at γ and δ will meet in the point

$$\left(a\frac{\cos\frac{1}{2}(\gamma+\delta)}{\cos\frac{1}{2}(\gamma-\delta)}, b\frac{\sin\frac{1}{2}(\gamma+\delta)}{\cos\frac{1}{2}(\gamma-\delta)}\right).$$

The co-ordinates of the middle point of the line joining these points of intersection are given by

$$x = \frac{a}{2}\,\frac{\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\gamma-\delta) + \cos\frac{1}{2}(\gamma+\delta)\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\gamma-\delta)\cos\frac{1}{2}(\alpha-\beta)},$$

$$y = \frac{b}{2}\,\frac{\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\gamma-\delta) + \sin\frac{1}{2}(\gamma+\delta)\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\gamma-\delta)\cos\frac{1}{2}(\alpha-\beta)}.$$

Therefore the line joining the centre of the ellipse to this point makes with the major axis an angle the tangent of which is

$$\frac{b}{a}\frac{\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\gamma-\delta)+\sin\frac{1}{2}(\gamma+\delta)\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\gamma-\delta)+\cos\frac{1}{2}(\gamma+\delta)\cos\frac{1}{2}(\alpha-\beta)},$$

which is equal to

$$\frac{b}{a}\frac{\sin(s-\alpha)+\sin(s-\beta)+\sin(s-\gamma)+\sin(s-\delta)}{\cos(s-\alpha)+\cos(s-\beta)+\cos(s-\gamma)+\cos(s-\delta)},$$

where $2s=\alpha+\beta+\gamma+\delta$.

The symmetry of the above result shews that the line joining the centre of the ellipse to one of the middle points of the diagonals of the quadrilateral will pass through the other two middle points.

EXAMPLES ON CHAPTER VI.

1. If SP, $S'P$ be the focal distances of a point P on an ellipse whose centre is C, and CD be the semi-diameter conjugate to CP; shew that $SP \cdot S'P = CD^2$.

2. The tangent at a point P of an ellipse meets the tangent at A, one extremity of the axis ACA', in the point Y; shew that CY is parallel to $A'P$, C being the centre of the curve.

3. A point moves so that the sum of the squares of its distances from two intersecting straight lines is constant. Prove that its locus is an ellipse, and find the eccentricity in terms of the angle between the lines.

4. P, Q are fixed points on an ellipse and R any other point on the curve; V, V' are the middle points of PR, QR, and VG, $V'G'$ are perpendicular to PR, QR respectively and meet the axis in G, G'. Shew that GG' is constant.

5. A series of ellipses are described with a given focus and corresponding directrix; shew that the locus of the extremities of their minor axes is a parabola.

6. PNP' is a double ordinate of an ellipse, and Q is any point on the curve; shew that, if QP, QP' meet the major axis in M, M' respectively, $CM \cdot CM' = CA^2$.

7. Lines are drawn through the foci of an ellipse perpendicular respectively to a pair of conjugate diameters and intersect in Q; shew that the locus of Q is a concentric ellipse.

8. The tangent at any point P of an ellipse cuts the equi-conjugate diameters in T, T'; shew that the triangles TCP, $T'CP$ are in the ratio of $CT^2 : CT'^2$.

9. If CQ be conjugate to the normal at P, then will CP be conjugate to the normal at Q.

10. If P, D be extremities of conjugate diameters of an ellipse, and PP', DD' be chords parallel to an axis of the ellipse; shew that PD' and $P'D$ are parallel to the equi-conjugates.

11. If P, D are extremities of conjugate diameters, and the tangent at P cut the major axis in T, and the tangent at D cut the minor axis in T'; shew that TT' will be parallel to one of the equi-conjugates.

12. QQ' is any chord of an ellipse parallel to one of the equi-conjugates, and the tangents at Q, Q' meet in T; shew that the circle QTQ' passes through the centre.

13. In the ellipse prove that the normal at any point is a fourth proportional to the perpendiculars on the tangent from the centre and from the two foci.

14. Two conjugate diameters of an ellipse are drawn, and their four extremities are joined to any point on a given circle whose centre is at the centre of the ellipse; shew that the sum of the squares of the lengths of these four lines is constant.

15. PNP' is a double ordinate of an ellipse whose centre is C, and the normal at P meets CP' in O; shew that the locus of O is an ellipse.

16. If the normal at any point P cut the major axis in G, shew that, for different positions of P, the locus of the middle point of PG will be an ellipse.

17. A, A' are the vertices of an ellipse, and P any point on the curve; shew that, if PN be perpendicular to AP and PM perpendicular to $A'P$, M, N being on the axis AA', then will MN be equal to the latus rectum of the ellipse.

18. Find the equation of the locus of a point from which two tangents can be drawn to an ellipse making angles θ_1, θ_2, with the axis-major such that (1) $\tan\theta_1 + \tan\theta_2$ is constant, (2) $\cot\theta_1 + \cot\theta_2$ is constant, and (3) $\tan\theta_1 \tan\theta_2$ is constant.

19. The line joining two extremities of any two diameters of an ellipse is either parallel or conjugate to the line joining two extremities of their conjugate diameters.

20. If P and D are extremities of conjugate diameters of an ellipse, shew that the tangents at P and D meet on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$, and that the locus of the middle point of PD is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}$.

21. A line is drawn parallel to the axis-minor of an ellipse midway between a focus and the corresponding directrix; prove that the product of the perpendiculars on it from the extremities of any chord passing through that focus is constant.

22. If the chord joining two points whose eccentric angles are α, β cut the major axis of an ellipse at a distance d from the centre, shew that $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{d-a}{d+a}$, where $2a$ is the length of the major axis.

23. If any two chords be drawn through two points on the axis-major of an ellipse equidistant from the centre, shew that $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2} = 1$, where α, β, γ, δ are the eccentric angles of the extremities of the chords.

24. If S, H be the foci of an ellipse and any point A be taken on the curve and the chords ASB, BHC, CSD, DHE... be drawn and the eccentric angles of $A, B, C, D, \ldots$ be $\theta_1, \theta_2, \theta_3, \theta_4 \ldots$, prove that $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \cot\frac{\theta_2}{2}\cot\frac{\theta_3}{2} = \tan\frac{\theta_3}{2}\tan\frac{\theta_4}{2} = \ldots$

25. Shew that the area of the triangle formed by the tangents at the points whose eccentric angles are α, β, γ respectively is $ab\tan\frac{1}{2}(\beta-\gamma)\tan\frac{1}{2}(\gamma-\alpha)\tan\frac{1}{2}(\alpha-\beta)$.

26. Prove that, if tangents be drawn to an ellipse at

points whose eccentric angles are ϕ_1, ϕ_2, ϕ_3, the radius of the circle circumscribing the triangle so formed is

$$\frac{pqr}{32ab} \sec\frac{\phi_2-\phi_3}{2} \sec\frac{\phi_3-\phi_1}{2} \sec\frac{\phi_1-\phi_2}{2};$$

p, q, r being the length of the diameters of the ellipse parallel to the sides of the triangle, and a, b the semi-axes of the ellipse.

27. From any point P on an ellipse straight lines are drawn through the foci S, H cutting the corresponding directrices in Q, R respectively; shew that the locus of the point of intersection of QH and RS is an ellipse.

28. If P, p be corresponding points on an ellipse and its auxiliary circle, centre C, and if CP be produced to meet the auxiliary circle in q; prove that the tangent at the point Q on the ellipse corresponding to q is perpendicular to Cp, and that it cuts off from Cp a length equal to CP.

29. If P, Q be the points of contact of perpendicular tangents to an ellipse, and p, q be the corresponding points on the auxiliary circle; shew that Cp, Cq are conjugate diameters of the ellipse.

30. From the centre C of two concentric circles two radii CQ, Cq are drawn equally inclined to a fixed straight line, the first to the outer circle, the second to the inner: prove that the locus of the middle point P of Qq is an ellipse, that PQ is the normal at P to this ellipse, and that Qq is equal to the diameter conjugate to CP.

31. If ω is the difference of the eccentric angles of two points on the ellipse the tangents at which are at right angles, prove that $ab \sin\omega = \lambda\mu$, where λ, μ are the semi-diameters parallel to the tangents at the points, and a, b are the semi-axes of the ellipse.

32. Two equal circles touch one another, find the locus of a point which moves so that the sum of the tangents from it to the two circles is constant.

33. Prove that the sum of the products of the perpendiculars from the two extremities of each of two conjugate diameters on any tangent to an ellipse is equal to the square of the perpendicular from the centre on that tangent.

34. Q is a point on the normal at any point P of an ellipse whose centre is C such that the lines CP, CQ make equal angles with the axis of the ellipse; shew that PQ is proportional to the diameter conjugate to CP.

35. If a pair of tangents to a conic be at right angles to one another, the product of the perpendiculars from the centre and the intersection of the tangents on the chord of contact is constant.

36. Find the locus of the middle points of chords of an ellipse which all pass through a fixed point.

37. If P be any point on an ellipse and any chord PQ cut the diameter conjugate to CP in R, then will $PQ\,.\,PR$ be equal to half the square on the diameter parallel to PQ.

38. Find the locus of the middle points of all chords of an ellipse which are of constant length.

39. Tangents at right angles are drawn to an ellipse; find the locus of the middle point of the chord of contact.

40. If three of the sides of a quadrilateral inscribed in an ellipse are parallel respectively to three given straight lines, shew that the fourth side will also be parallel to a fixed straight line.

41. The area of the parallelogram formed by the tangents at the ends of any pair of diameters of an ellipse varies inversely as the area of the parallelogram formed by joining the points of contact.

42. If at the extremities P, Q of any two diameters CP, CQ of an ellipse, two tangents Pp, Qq be drawn cutting each other in T and the diameters produced in p, and q, then the areas of the triangles TQp, TPq will be equal.

43. From the point O two tangents OP, OQ are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; shew that the area of the triangle CPQ is equal to

$$\frac{a^2b^2\sqrt{b^2h^2+a^2k^2-a^2b^2}}{b^2h^2+a^2k^2},$$

and the area of the quadrilateral $OPCQ$ is equal to

$$(b^2h^2+a^2k^2-a^2b^2)^{\frac{1}{2}},$$

C being the centre of the ellipse, and h, k the co-ordinates of O.

44. TP, TQ are tangents to an ellipse whose centre is C, shew that the area of the quadrilateral $CPTQ$ is $ab \tan \frac{1}{2}(\phi - \phi')$; where a, b are the semi-axes of the ellipse, and ϕ, ϕ' are the eccentric angles of P and Q.

45. PCP' is a diameter of an ellipse and QCQ' is the corresponding diameter of the auxiliary circle; shew that the area of the parallelogram formed by the tangents at P, P', Q, Q' is $\frac{8a^2b}{(a-b)\sin 2\phi}$, where ϕ is the eccentric angle of P.

46. A parallelogram circumscribes a circle, and two of the angular points are on fixed straight lines parallel to one another and equidistant from the centre; shew that the other two are on an ellipse of which the circle is the minor auxiliary circle.

47. Two fixed conjugate diameters of an ellipse are met in the points P, Q respectively by two lines OP, OQ which pass through a fixed point O and are parallel to any other pair of conjugate diameters; shew that the locus of the middle point of PQ is a straight line.

48. If from any point O in the plane of an ellipse the perpendiculars OM, ON be drawn on the equal conjugate diameters, the direction OP of the diagonal of the parallelogram $MONP$ will be perpendicular to the polar of O.

49. Three points A, P, B are taken on an ellipse whose centre is C. Parallels to the tangents at A and B drawn through P meet CB and CA respectively in the points Q and R. Prove that QR is parallel to the tangent at P.

50. Find the locus of the point of intersection of normals at two points on an ellipse which are extremities of conjugate diameters.

51. Normals to an ellipse are drawn at the extremities of a chord parallel to one of the equi-conjugate diameters; prove that they intersect on a diameter perpendicular to the other equi-conjugate.

52. If normals be drawn at the extremities of any focal chord of an ellipse, a line through their intersection parallel to the axis-major will bisect the chord.

53. If a length PQ be taken in the normal at any point P of an ellipse whose centre is C, equal in length to the semi-diameter which is conjugate to CP, shew that Q lies on one or other of two circles.

54. Shew that, if ϕ be the angle between the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ drawn from the point (x', y'), then will $\tan\phi\,(x'^2 + y'^2 - a^2 - b^2) = 2ab\sqrt{\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1}$.

55. TP, TQ are the tangents drawn from an external point (x, y) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$; shew that, if S be a focus,

$$\frac{ST^2}{SP \,.\, SQ} = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

56. If two tangents to an ellipse from a point T intersect at an angle ϕ, shew that $ST \,.\, HT \cos\phi = CT^2 - a^2 - b^2$, where C is the centre of the ellipse and S, H the foci.

57. If the perpendicular from the centre C of an ellipse on the tangent at any point P meet the focal distance SP, produced if necessary, in R; the locus of R will be a circle.

58. If two concentric ellipses be such that the foci of one lie on the other, and if e, e' be their eccentricities, shew that their axes are inclined at an angle $\cos^{-1}\frac{\sqrt{e^2 + e'^2 - 1}}{ee'}$.

59. Shew that the angle which a diameter of an ellipse subtends at either end of the axis-major is supplementary to that which the conjugate diameter subtends at the end of the axis-minor.

60. If θ, θ' be the angles subtended by the axis major of an ellipse at the extremities of a pair of conjugate diameters, shew that $\cot^2\theta + \cot^2\theta'$ is constant.

61. If the distance between the foci of an ellipse subtend angles 2θ, $2\theta'$ at the ends of a pair of conjugate diameters, shew that $\tan^2\theta + \tan^2\theta'$ is constant.

62. If λ, λ' be the angles which any two conjugate diameters subtend at any fixed point on an ellipse, prove that $\cot^2\lambda + \cot^2\lambda'$ is constant.

63. Shew that pairs of conjugate diameters of an ellipse are cut in involution by any straight line.

64. A triangle whose sides touch an ellipse and enclose it, is a minimum; shew that each side of the triangle touches at its middle point, and that the triangle formed by joining the points of contact is a maximum.

65. A, B, C, D are four fixed points on an ellipse, and P any other point on the curve; shew that the product of the perpendiculars from P on AB and CD bears a constant ratio to the product of the perpendiculars from P on BC and DA.

66. Find the locus of the point of intersection of two normals to an ellipse which are perpendicular to one another.

67. Find the equation of the locus of the point of intersection of the tangent at one end of a focal chord of an ellipse with the normal at the other end.

68. Two straight lines are drawn parallel to the axis-major of an ellipse at a distance $\frac{ab}{\sqrt{a^2 - b^2}}$ from it; prove that the part of any tangent intercepted between them is divided by the point of contact into two parts which subtend equal angles at the centre.

69. PG is the normal to an ellipse at P, G being in the major axis, GP is produced outwards to Q so that $PQ = GP$; shew that the locus of Q is an ellipse whose eccentricity is $\frac{a^2 - b^2}{a^2 + b^2}$, and find the equation of the locus of the intersection of the tangents at P and Q.

CHAPTER VII.

THE HYPERBOLA.

Definition. The *Hyperbola* is the locus of a point which moves so that its distance from a fixed point, called the focus, bears a constant ratio, which is greater than unity, to its distance from a fixed straight line, called the directrix.

139. *To find the equation of an hyperbola.*

Let S be the focus and ZM the directrix.

Draw SZ perpendicular to the directrix.

Divide ZS in A so that $SA : AZ =$ given ratio $= e : 1$ suppose. Then A is a point on the curve.

There will also be a point A' in SZ produced such that

$$SA' : A'Z :: e : 1.$$

Let C be the middle point of AA', and let $AA' = 2a$.

Then $\quad SA = e \,.\, AZ$, and $SA' = e \,.\, A'Z$;

$$\therefore\ SA + SA' = e\,(AZ + A'Z);$$

$$\therefore\ 2SC = 2e \,.\, AC;$$

$$\therefore\ CS = ae \quad \dots\dots\dots\dots\dots\dots\dots\text{(i)}.$$

Also $\quad SA' - SA = e\,(A'Z - AZ)$,

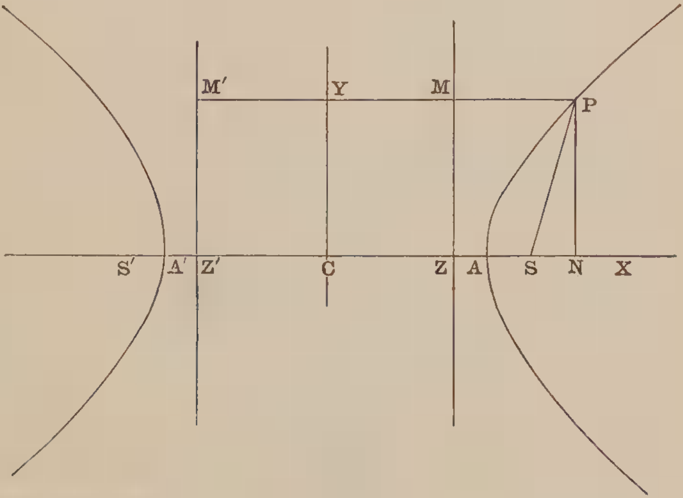
or $\quad AA' = e\,(AA' - 2AZ);$

$$\therefore\ AC = e \, . \, ZC,$$

or $$CZ = \frac{a}{e} \quad \text{..........................(ii)}.$$

Now let C be taken as origin, CA as the axis of x, and a line perpendicular to CA as the axis of y.

Let P be any point on the curve, and let its co-ordinates be x, y.



Then, in the figure

$$SP^2 = e^2 PM^2;$$

$$\therefore\ SN^2 + NP^2 = e^2 ZN^2.$$

Now $$SN = CN - CS = x - ae,$$

and $$ZN = CN - CZ = x - \frac{a}{e};$$

$$\therefore\ (x - ae)^2 + y^2 = e^2 \left(x - \frac{a}{e}\right)^2,$$

or $$y^2 + x^2 (1 - e^2) = a^2 (1 - e^2),$$

or $$\frac{x^2}{a^2} + \frac{y^2}{a^2 (1 - e^2)} = 1 \quad \text{...............(iii)}.$$

Since e is greater than unity $a^2(1-e^2)$ is *negative;* if we put $-b^2$ for $a^2(1-e^2)$, the equation takes the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ ....(iv).}$$

The *latus rectum* is the chord through the focus parallel to the directrix. To find its length we must put $x=ae$ in equation (iv).

Then $y^2=b^2(e^2-1)=\dfrac{b^4}{a^2}$, since $b^2=a^2(e^2-1)$;

so that the length of the semi-latus rectum is $\dfrac{b^2}{a}$.

140. In equation (iv) [Art. 139] x^2 cannot be less than a^2, for otherwise y^2 would be negative.

Hence no part of the curve lies between

$$x=-a \text{ and } x=a.$$

If x be greater than a, y^2 will be positive; and for any particular value of x there will be two equal and opposite values of y. Therefore the axis of x divides the curve into two similar and equal parts.

For any value of y, x^2 is positive; and for any particular value of y there will be two equal and opposite values of x. Therefore the axis of y divides the curve into two similar and equal parts. From this it follows that if on the axis of x the points S', Z' be taken such that $CS'=SC$, and $CZ'=ZC$, the point S' will also be a focus of the curve, and the line through Z' perpendicular to CZ' will be the corresponding directrix.

If (x', y') be any point on the curve, it is clear that the point $(-x', -y')$ will also be on the curve. But the points (x', y') and $(-x', -y')$ are on a straight line through the origin and are equidistant from the origin. Hence the origin bisects every chord which passes through it, and is therefore called the *centre* of the curve.

From equation (iv) [Art. 139] it is clear that if x^2 be greater than a^2, y^2 will be positive, and will get larger and larger as x^2 becomes larger and larger, and there is no limit to this increase of x and y. The curve is therefore shaped somewhat as in the figure to Art. 139, and consists of two infinite branches.

AA' is called the *transverse axis* of the hyperbola. The line through C perpendicular to AA' does not meet the curve in real points; but, if B, B' be the points on this line such that $BC = CB' = b$, the line BB' is called the *conjugate* axis.

141. *To find the focal distances of any point on an hyperbola.*

In the figure to Art. 139, since $SP = ePM$, we have

$$SP = eZN = e\,(CN - CZ) = e\left(x - \frac{a}{e}\right) = ex - a:$$

also $$S'P = e\,.\,PM' = e(CN + Z'C) = e\left(x + \frac{a}{e}\right) = ex + a;$$

$$\therefore\ S'P - SP = 2a.$$

142. The polar equation of the hyperbola referred to the centre as pole will be found by writing $r\cos\theta$ for x, and $r\sin\theta$ for y in the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The equation will therefore be

$$\frac{r^2\cos^2\theta}{a^2} - \frac{r^2\sin^2\theta}{b^2} = 1,$$

or $$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} \text{ ....(i).}$$

The equation (i) can be written in the form

$$\frac{1}{r^2} = \frac{1}{a^2} - \left(\frac{1}{a^2} + \frac{1}{b^2}\right)\sin^2\theta \text{ ....(ii).}$$

We see from (ii) that $\frac{1}{r^2}$ is greatest, and therefore r is least, when θ is zero. As θ increases, $\frac{1}{r^2}$ diminishes, and is zero when $\sin^2\theta = \frac{b^2}{a^2 + b^2}$; so that for this value of θ,

r is infinite. If $\sin^2\theta$ be greater than $\frac{b^2}{a^2+b^2}$, $\frac{1}{r^2}$ will be negative, so that a radius vector which makes with the axis an angle greater than $\sin^{-1}\frac{b}{\sqrt{(a^2+b^2)}}$ does not meet the curve in real points.

143. Most of the results obtained in the preceding chapter hold good for the hyperbola, and in the proofs there given it is only necessary to change the sign of b^2. We shall therefore only enumerate them.

Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

(i) The line $y=mx+\sqrt{(a^2m^2-b^2)}$ is a tangent for all values of m [Art. 113].

(ii) The equation of the tangent at (x', y') is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1. \quad \text{[Art. 114.]}$$

(iii) The equation of the polar of (x', y') is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1. \quad \text{[Art. 118.]}$$

(iv) The equation of the normal at (x', y') is

$$\frac{x-x'}{\frac{x'}{a^2}}=\frac{y-y'}{\frac{y'}{-b^2}}. \quad \text{[Art. 116.]}$$

(v) The line $lx+my=n$ will touch the curve, if $a^2l^2-b^2m^2=n^2$ [Art. 115].

(vi) The line $x\cos\alpha+y\sin\alpha=p$ will touch the curve, if $p^2=a^2\cos^2\alpha-b^2\sin^2\alpha$ [Art. 115].

(vii) The equation of the director-circle of the hyperbola is $x^2+y^2=a^2-b^2$ [Art. 120].

The director-circle is clearly imaginary when a is less than b, and reduces to a point when $a=b$.

(viii) The geometrical propositions proved in Art. 125 are also true for the hyperbola.

(ix) The locus of the middle points of all chords of the hyperbola which are parallel to $y = mx$ is the straight line $y = m'x$, where $mm' = \frac{b^2}{a^2}$ [Art. 127].

144. The lines $y = mx$, $y = m'x$ are conjugate if

$$mm' = \frac{b^2}{a^2}.$$

These two diameters meet the curve in points whose abscissæ are given by the equations

$$x^2\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right) = 1, \text{ and } x^2\left(\frac{1}{a^2} - \frac{m'^2}{b^2}\right) = 1.$$

The first equation gives real values of x if m be less than $\frac{b}{a}$, and the second gives real values if m' be less than $\frac{b}{a}$. But, since $mm' = \frac{b^2}{a^2}$, m and m' cannot both be less than $\frac{b}{a}$, nor both be greater.

Therefore, of two conjugate diameters of an hyperbola one meets the curve in *real* points, and the other in *imaginary* points.

The two conjugate diameters are *coincident* if $m = \pm\frac{b}{a}$.

145. Let P, D be extremities of a pair of conjugate diameters; let the co-ordinates of P be x', y', and the co-ordinates of D be x'', y''. We know from Art. 144 that if one of these two points be real the other will be imaginary.

The equations of CP and CD are

Omit.

$$\frac{y}{y'} = \frac{x}{x'} \text{ and } \frac{y}{y''} = \frac{x}{x''}.$$

Hence, from (ix) Art. 143, we have

$$\frac{x'x''}{a^2} - \frac{y'y''}{b^2} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

whence
$$\frac{x'^2x''^2}{a^4} = \frac{y'^2y''^2}{b^4},$$

or, since (x', y') and (x'', y'') are both on the curve,

$$\frac{x''^2}{a^2}\left(1 + \frac{y'^2}{b^2}\right) = \left(\frac{x''^2}{a^2} - 1\right)\frac{y'^2}{b^2},$$

or
$$\frac{x''^2}{a^2} = -\frac{y'^2}{b^2};$$

$$\therefore\; x'' = \pm\frac{a}{b}y'\sqrt{-1} \;\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

and $\therefore$ from (i)
$$y'' = \pm\frac{b}{a}x'\sqrt{-1} \;\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

From (ii) and (iii) we have

$$\begin{aligned} CP^2 + CD^2 &= x'^2 + y'^2 - \frac{a^2}{b^2}y'^2 - \frac{b^2}{a^2}x'^2 \\ &= a^2\left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right) - b^2\left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right), \\ &= a^2 - b^2. \end{aligned}$$

So that, as in the case of the ellipse, the *sum* of the squares of two conjugate diameters is constant.

146. *Definition.* An *asymptote* is a straight line which meets a curve in *two* points at infinity, but which is not altogether at infinity.

To find the asymptotes of an hyperbola.

To find the abscissæ of the points where the straight line $y = mx + c$ cuts the hyperbola, we have the equation

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1,$$

or
$$x^2\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right) - \frac{2mc}{b^2}x - \frac{c^2}{b^2} - 1 = 0 \;\ldots\ldots\ldots\text{(i)}.$$

Both roots of the equation (i) will be infinite if the coefficients of x^2 and of x are both zero; that is, if

$$\frac{1}{a^2}-\frac{m^2}{b^2}=0, \text{ and } mc=0.$$

Hence we must have $c=0$, and $m=\pm\frac{b}{a}$.

The hyperbola $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

has therefore two real asymptotes whose equations are $y=\pm\frac{b}{a}x$; or, expressed in one equation,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \text{..................(ii)}.$$

Draw lines through B, B' parallel to the transverse axis, and through A, A' parallel to the conjugate axis; then we see from (ii) that the asymptotes are the diagonals of the rectangle so formed.

The ellipse has no real points at infinity, and therefore the asymptotes of an ellipse are imaginary.

147. Any straight line parallel to an asymptote will meet the curve in *one* point at infinity.

For, one root of the equation (i) Art. 146 will be infinite, if the coefficient of x^2 is zero. This will be the case if $m=\pm\frac{b}{a}$. So that the line $y=\pm\frac{b}{a}x+c$ meets the curve in *one* point at infinity whatever the value of c may be.

148. The equation of the hyperbola which has BB' for its transverse axis and AA' for its conjugate axis is

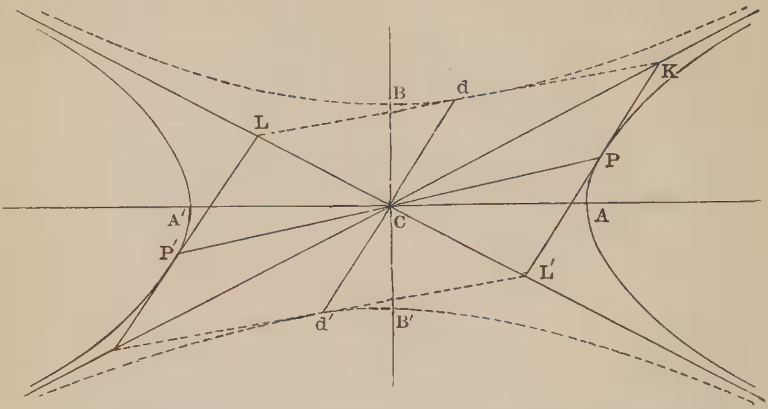
$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{..................(i)}.$$

This hyperbola and the original hyperbola, whose equation is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{..................(ii)},$$

are said to be *conjugate* to one another.

We append some properties of a pair of conjugate hyperbolas.



(1) The two hyperbolas have the same asymptotes.

(2) If two diameters be conjugate with respect to one of the hyperbolas, they will be conjugate with respect to the other.

This follows from the condition in (ix) Article 143.

(3) The equations of the hyperbolas (ii) and (i) can [Art. 142] be written in the forms

$$\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2},$$

$$-\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}.$$

It is clear that if, for any value of θ, r^2 is positive for one curve it is negative for the other.

Hence every diameter meets one curve in real points and the other in imaginary points; moreover the lengths of semi-diameters of the two curves are, for all values of θ, connected by the relation $r_1{}^2=-r_2{}^2$.

(4) If two conjugate diameters cut the curves (ii) and (i) in P and d respectively, then $CP^2-Cd^2=a^2-b^2$.

Let x', y' be the co-ordinates of P, and x'', y'' the co-ordinates of d.

Then the equations of CP and Cd are

$$\frac{x}{x'} - \frac{y}{y'} = 0, \text{ and } \frac{x}{x''} - \frac{y}{y''} = 0.$$

The condition for conjugate diameters, viz. $mm' = \frac{b^2}{a^2}$,

gives $$\frac{x'x''}{a^2} - \frac{y'y''}{b^2} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)},$$

or $$\frac{x'^2x''^2}{a^4} = \frac{y'^2y''^2}{b^4}.$$

And, since (x', y') is on (ii), and (x'', y'') on (i), we

have $$\frac{x'^2}{a^2}\left(\frac{y''^2}{b^2} - 1\right) = \frac{y''^2}{b^2}\left(\frac{x'^2}{a^2} - 1\right),$$

or $$\frac{x'^2}{a^2} = \frac{y''^2}{b^2};$$

$$\therefore \frac{y''}{b} = \pm \frac{x'}{a} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)},$$

and, ∴ from (iii), $$\frac{x''}{a} = \pm \frac{y'}{b} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}.$$

Hence $$CP^2 - Cd^2 = x'^2 + y'^2 - x''^2 - y''^2$$

$$= x'^2 + y'^2 - \frac{a^2}{b^2}y'^2 - \frac{b^2}{a^2}x'^2$$

$$= (a^2 - b^2)\left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right);$$

$$\therefore CP^2 - Cd^2 = a^2 - b^2\text{*}.$$

(5) The parallelogram formed by the tangents at P, P', d, d' is of constant area.

The parallelogram is equal to $4CP \,.\, Cd \sin PCd$, or equal to $4Cd \,.\, CF$, where CF is the perpendicular from C on the tangent at P.

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* CP and Cd must not be looked upon as conjugate semi-diameters, since the points P and d are not on the same hyperbola. The line dCd' cuts the original hyperbola in two imaginary points; and if these points be D, D', we see from (3) that $CD^2 = -Cd^2$.

Now the equation of the tangent at P is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1;$$

$$\therefore\ CF^2 = \frac{1}{\dfrac{x'^2}{a^4} + \dfrac{y'^2}{b^4}}.$$

And $$Cd^2 = \frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2 = a^2b^2\left(\frac{x'^2}{a^4} + \frac{y'^2}{b^4}\right).$$

Hence $$Cd\ .\ CF = ab.$$

(6) The asymptotes bisect Pd and Pd'.

If x, y be the co-ordinates of the middle point of Pd, then

$$2x = x' + x'',\ \text{and}\ 2y = y' + y'';$$

$$\therefore\ \frac{x}{y} = \frac{x' + x''}{y' + y''} = \frac{x' \pm \dfrac{a}{b}y'}{y' \pm \dfrac{b}{a}x'} = \pm\frac{a}{b};$$

therefore the middle points of Pd and of Pd' are on one or other of the lines

$$\frac{x}{a} = \pm\frac{y}{b}.$$

Also, since $CPKd$ is a parallelogram CK bisects Pd or Pd', and therefore is one of the asymptotes, so that the tangents at D, D' meet those at d, d' on the asymptotes.

(7) The equations of the polars of (x', y') with respect to the hyperbolas (ii) and (i) respectively are

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1,\ \text{and}\ -\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1.$$

Hence the polars of any point with respect to the two curves are parallel to one another and equidistant from the centre.

If (x', y') be any point P on (ii), then its polar with respect to (i) is

$$-\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,\ \text{or}\ \frac{x(-x')}{a^2} - \frac{y(-y')}{b^2} = 1.$$

But the last equation is the tangent to (ii) at the point $(-x', -y')$, which is the other extremity of the diameter through P.

Hence, if from any point on an hyperbola the tangents PQ, PQ' be drawn to the conjugate hyperbola, the line QQ' will touch the original hyperbola at the other end of the diameter through P.

149. *To find the equation of an hyperbola referred to any pair of conjugate diameters as axes.*

The equation of the hyperbola referred to its transverse and conjugate axes is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Since the origin is unaltered we substitute for x, y expressions of the form $lx + my$, $l'x + m'y$ in order to obtain the transformed equation [Art. 51].

The equation of the hyperbola will therefore be of the form

$$Ax^2 + 2Hxy + By^2 = 1 \text{ .... (i).}$$

By supposition the axis of x bisects the chords parallel to the axis of y. Therefore for *any* particular value of x the two values of y found from (i) must be equal and opposite. Hence $H = 0$; the equation will therefore be of the form

$$Ax^2 + By^2 = 1 \text{ .... (ii).}$$

Of the two semi-conjugate diameters one is real and the other imaginary. If their lengths be a' and $\sqrt{-1}\,b'$; since these are the intercepts on the axes of x and y respectively, we obtain from (ii)

$$Aa'^2 = 1 = -Bb'^2.$$

Hence the required equation is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1 \text{ .... (iii).}$$

150. Since the equation of the curve is of the same form as before, all investigations in which it was not assumed

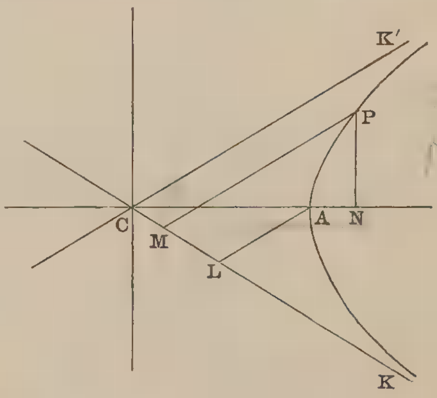
that the axes were at right angles to one another still hold good. For example (i), (ii), (iii), (v) and (ix) of Art. 143 require no change. Art. 146 will also apply without change, so that the equation of the asymptotes of the hyperbola whose equation is (ii) is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 0.$$

151. *To find the equation of an hyperbola when referred to its asymptotes as axes of co-ordinates.*

Let the asymptotes be the lines CK, CK' in the figure, and let the angle $ACK' = \alpha$, so that $\tan \alpha = \dfrac{b}{a}$.

Let P be any point (x, y) of the curve, and let x', y' be the co-ordinates of P when referred to CK, CK'. Draw PM parallel to CK' to meet CK in M, and draw PN perpendicular to the transverse axis.



Then $\quad CM = x'$, $MP = y'$, $CN = x$, $NP = y$.

Now $\quad CN = CM \cos \alpha + MP \cos \alpha$,

or $$x = (x' + y') \cos \alpha \quad\text{...............(i).}$$

Also $\quad NP = MP \sin \alpha - CM \sin \alpha$,

or $$y = (y' - x') \sin \alpha \quad\text{.............(ii).}$$

Hence, by substituting in the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

we obtain

$$\frac{\cos^2\alpha\,(x' + y')^2}{a^2} - \frac{\sin^2\alpha\,(y' - x')^2}{b^2} = 1 \ldots\ldots(\text{iii}).$$

But $\tan\alpha = \frac{b}{a}$, therefore $\frac{\sin^2\alpha}{b^2} = \frac{\cos^2\alpha}{a^2} = \frac{1}{a^2 + b^2}$.

Hence, suppressing the accents, we have from (iii)

$$4xy = a^2 + b^2,$$

which is the required equation.

The equation of the conjugate hyperbola, when referred to the asymptotes, will be

$$4xy = -(a^2 + b^2).$$

152. The equations of an hyperbola, of the asymptotes, and of the conjugate hyperbola are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \text{ and } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

respectively.

If the axes of co-ordinates be changed in any manner, we should, in order to obtain the new equations, have to make the same substitutions in all three cases.

Hence, for all positions of the axes of co-ordinates, the equations of an hyperbola and of the conjugate hyperbola will only differ from the equation of the asymptotes by constants, and the two constants will be equal and opposite for the two hyperbolas.

153. *To find the equation of the tangent at any point of the hyperbola whose equation is* $4xy = a^2 + b^2$.

The equation of the line joining the two points (x', y'), (x'', y'') is

$$\frac{y - y'}{y'' - y'} = \frac{x - x'}{x'' - x'} \ldots\ldots\ldots\ldots\ldots\ldots(\text{i}).$$

But, since the points (x', y') and (x'', y'') are on the curve,

$$x'y' = \frac{a^2+b^2}{4} = x''y'';$$

$$\therefore\ y'' - y' = \frac{x'y'}{x''} - y',$$

or
$$\frac{y''-y'}{y'} = \frac{x'-x''}{x''} \quad\text{..................(ii).}$$

From (i) and (ii) we have

$$\frac{y-y'}{y'} = -\frac{x-x'}{x''}.$$

The equation of the tangent at (x', y') is therefore

$$\frac{y-y'}{y'} + \frac{x-x'}{x'} = 0,$$

or
$$\frac{x}{x'} + \frac{y}{y'} = 2 \quad\text{......................(iii).}$$

From (iii) we see that the intercepts on the axes are $2x'$ and $2y'$.

Hence the portion of the tangent intercepted by the asymptotes is bisected at the point of contact.

The area of the triangle cut off from the asymptotes by any tangent is from (iii) equal to $2x'y' \sin\omega$; or, since $4x'y' = a^2 + b^2$, and $\sin\omega = \frac{2ab}{a^2+b^2}$, the area of the triangle is equal to ab.

154. When the angle between the asymptotes of an hyperbola is a right angle it is called a *rectangular hyperbola.*

The angle between the asymptotes is equal to $2\tan^{-1}\frac{b}{a}$, and therefore when the angle is a right angle we have $b = a$. On this account the curve is sometimes called an *equilateral* hyperbola.

155. *The asymptotes and any pair of conjugate diameters of an hyperbola form a harmonic pencil.*

The tangent at the extremity of any diameter of an hyperbola is parallel to the conjugate diameter; also [Art. 153], the portion of the tangent intercepted by the asymptotes is bisected at the point of contact. Hence [Art. 55] the pencil formed by the asymptotes and a pair of conjugate diameters is harmonic.

156. We may, as in the case of the ellipse, express the co-ordinates of any point on the hyperbola in terms of a single parameter. We may put $x = a \sec\theta$, and $y = b \tan\theta$, since for all values of θ, $\sec^2\theta - \tan^2\theta = 1$.

If PN be the ordinate of any point P on the curve, and NQ be the tangent from N to the auxiliary circle; then $CN = a \sec ACQ$. Hence ACQ is the angle θ.

157. The equation of an ellipse or hyperbola referred to a vertex as origin is found by writing $x - a$ for x in the equation referred to the centre as origin. The equation will therefore be

$$\frac{(x-a)^2}{a^2} \pm \frac{y^2}{b^2} = 1,$$

or
$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} - \frac{2x}{a} = 0 \text{.................(i).}$$

Now, if the distance from the vertex to the nearer focus remain fixed (d suppose), and the eccentricity become unity, the curve will become a parabola of latus rectum $4d$.

The equation of the parabola can be deduced from (i). For, since $a(1-e) = d$, a must be infinite when $e = 1$.

Also $a(1-e^2) = d(1+e) = 2d$; therefore $\frac{b^2}{a} = 2d$.

Hence, from (i)

$$\frac{x^2}{a} \pm \frac{y^2}{2d} - 2x = 0,$$

or, since a is infinite,

$$y^2 = \pm 4dx.$$

The parabola therefore is a limiting form of an ellipse or of an hyperbola, the latus rectum of which is finite, but the major and minor axes are infinite. The centre and the second focus are at infinity.

It is a very instructive exercise for the student to deduce the properties of a parabola from those of an ellipse or hyperbola.

158. Let the focus of a conic be *on* the directrix.

Take the focus as origin, and let the directrix be the axis of y; then the equation of the conic will be

$$x^2 + y^2 = e^2x^2,$$

or

$$x^2(1-e^2) + y^2 = 0.$$

This equation represents *two straight lines* which are real if e be greater than unity, coincident if e be equal to unity, and imaginary if e be less than unity.

Hence we must not only consider as conics an ellipse, a parabola, and an hyperbola, but also two real or imaginary straight lines.

It should be noticed that the directrix of a circle is at an infinite distance; also that the foci and directrices of two parallel straight lines are all at infinity.

EXAMPLES ON CHAPTER VII.

1. AOB, COD are two straight lines which bisect one another at right angles; shew that the locus of a point which moves so that $PA \,.\, PB = PC \,.\, PD$ is a rectangular hyperbola.

2. If a straight line cut an hyperbola in Q, Q' and its asymptotes in R, R', shew that the middle point of QQ' will be the middle point of RR'.

3. A straight line has its extremities on two fixed straight lines and passes through a fixed point; find the locus of the middle point of the line.

4. A straight line has its extremities on two fixed straight lines and cuts off from them a triangle of constant area; find the locus of the middle point of the line.

5. OA, OB are fixed straight lines, P any point, and PM, PN the perpendiculars from P on OA, OB; find the locus of P if the quadrilateral $OMPN$ be of constant area.

6. The distance of any point from the centre of a rectangular hyperbola varies inversely as the perpendicular distance of its polar from the centre.

7. PN is the ordinate of a point P on an hyperbola, PG is the normal meeting the axis in G; if NP be produced to meet the asymptote in Q, prove that QG is at right angles to the asymptote.

8. If e, e' be the eccentricities of an hyperbola and of the conjugate hyperbola, then will $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.

9. The two straight lines joining the points in which any two tangents to an hyperbola meet the asymptotes are parallel to the chord of contact of the tangents and are equidistant from it.

10. Prove that the part of the tangent at any point of an hyperbola intercepted between the point of contact and the transverse axis is a harmonic mean between the lengths of the perpendiculars drawn from the foci on the normal at the same point.

11. If through any point O a line OPQ be drawn parallel to an asymptote of an hyperbola cutting the curve in P and the polar of O in Q, shew that P is the middle point of OQ.

12. A parallelogram is constructed with its sides parallel to the asymptotes of an hyperbola, and one of its diagonals is a chord of the hyperbola; shew that the direction of the other will pass through the centre.

13. A, A' are the vertices of a rectangular hyperbola, and P is any point on the curve; shew that the internal and external bisectors of the angle APA' are parallel to the asymptotes.

14. A, A' are the extremities of a fixed diameter of a circle and P, P' are the extremities of any chord perpendicular to this diameter; shew that the locus of the point of intersection of AP and $A'P'$ is a rectangular hyperbola.

15. Shew that the co-ordinates of the point of intersection of two tangents to an hyperbola referred to its asymptotes as axes are harmonic means between the co-ordinates of the points of contact.

16. From any point of one hyperbola tangents are drawn to another which has the same asymptotes; shew that the chord of contact cuts off a constant area from the asymptotes.

17. The straight lines drawn from any point of an equilateral hyperbola to the extremities of any diameter are equally inclined to the asymptotes.

18. The locus of the middle points of normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2x^2y^2$.

19. Shew that the line $x = 0$ is an asymptote of the hyperbola $2xy + 3x^2 + 4x = 9$.

What is the equation of the other asymptote?

20. Find the asymptotes of $xy - 3x - 2y = 0$.

What is the equation of the conjugate hyperbola?

21. Shew that in an hyperbola the ratio of the tangents of half the angles which the radii vectores from the foci to a point on the curve make with the axis, is constant.

22. A circle intersects an hyperbola in four points; prove that the product of the distances of the four points of intersection from one asymptote is equal to the product of their distances from the other.

23. Shew that if a rectangular hyperbola cut a circle in four points the centre of mean position of the four points is midway between the centres of the two curves.

24. If four points be taken on a rectangular hyperbola such that the chord joining any two is perpendicular to the chord joining the other two, and if α, β, γ, δ be the inclinations

to either asymptote of the straight lines joining these points respectively to the centre; prove that $\tan\alpha\tan\beta\tan\gamma\tan\delta = 1$.

25. A series of chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are tangents to the circle described on the straight line joining the foci of the hyperbola as diameter; shew that the locus of their poles with respect to the hyperbola is $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2 + b^2}$.

26. If two straight lines pass through fixed points, and the bisector of the angle between them is always parallel to a fixed line, prove that the locus of the point of intersection of the lines is a rectangular hyperbola.

27. Shew that pairs of conjugate diameters of an hyperbola are cut in involution by any straight line.

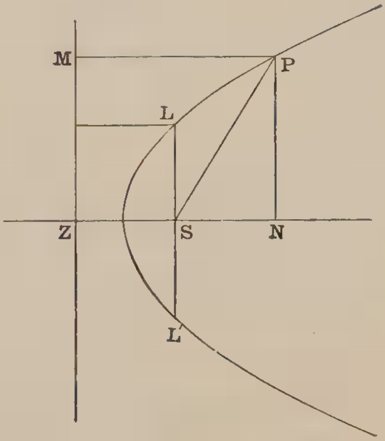
28. The locus of the intersection of two equal circles, which are described on two sides AB, AC of a triangle as chords, is a rectangular hyperbola, whose centre is the middle point of BC, and which passes through A, B, C.

CHAPTER VIII.

POLAR EQUATION OF A CONIC, THE FOCUS BEING THE POLE.

159. *To find the polar equation of a conic, the focus being the pole.*

Let S be the focus and ZM the directrix of the conic, and let the eccentricity be e.



Draw SZ perpendicular to the directrix, and let SZ be taken for initial line.

Let LSL' be the latus rectum, then $e . SZ = SL = l$ suppose.

Let the co-ordinates of any point P on the curve be r, θ. Let PM, PN be perpendicular respectively to the directrix and to SZ.

Then we have

$$SP = e \,.\, PM = e \,.\, NZ = e \,.\, NS + e \,.\, SZ,$$

or
$$r = e \,.\, r \cos(\pi - \theta) + l;$$

$$\therefore \quad \frac{l}{r} = 1 + e \cos\theta.$$

If the axis of the conic make an angle α with the initial line the equation of the curve will be

$$\frac{l}{r} = 1 + e \cos(\theta - \alpha).$$

For in this case SP makes with SZ an angle $\theta - \alpha$.

160. If r, θ be the co-ordinates of any point on the directrix, then

$$r \cos\theta = SZ = \frac{l}{e};$$

therefore *the equation of the directrix* is

$$\frac{l}{r} = e \cos\theta.$$

The equation of the directrix of $\frac{l}{r} = 1 + e \cos \overline{\theta - \alpha}$ is

$$\frac{l}{r} = e \cos(\theta - \alpha).$$

161. *To shew that in any conic the semi-latus rectum is a harmonic mean between the segments of any focal chord.*

If PSP' be the focal chord, and the vectorial angle of P be θ, that of P' will be $\theta + \pi$.

Hence, if $SP = r$, and $SP' = r'$, we have

$$\frac{l}{r} = 1 + e \cos\theta, \text{ and } \frac{l}{r'} = 1 + e \cos(\theta + \pi);$$

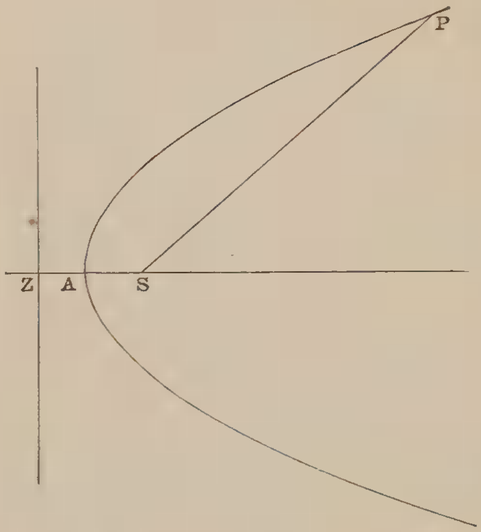
$$\therefore \quad \frac{l}{r} + \frac{l}{r'} = 2.$$

Hence
$$\frac{1}{r} + \frac{1}{r'} = \frac{2}{l}.$$

162. To trace the conic $\frac{l}{r} = 1 + e\cos\theta$ from its equation.

(1) Let $e = 1$, then the curve is a parabola, and the equation becomes

$$\frac{l}{r} = 1 + \cos\theta.$$



At the point A, where the curve cuts the axis,

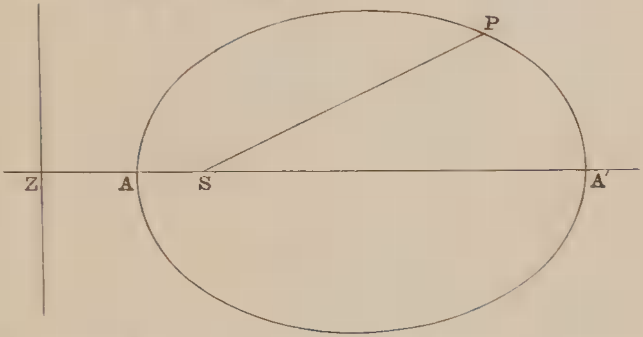
$$\theta = 0 \text{ and } r = \frac{l}{2}.$$

As the angle θ increases, $(1 + \cos\theta)$ decreases, that is $\frac{l}{r}$ decreases, and therefore r increases: and r increases without limit until $\theta = \pi$, when r is infinite. As θ increases beyond π, $1 + \cos\theta$ increases continuously, and therefore r decreases continuously until when $\theta = 2\pi$ it again becomes equal to $\frac{l}{2}$. The curve therefore is as in the figure going to an infinite distance in the direction AS.

(2) Let e be less than unity, then the curve is an ellipse.

At the point A, $\theta = 0$, and $r = \frac{l}{1+e}$.

As θ increases $\cos\theta$ decreases, and therefore $\frac{l}{r}$ decreases, that is r increases, until $\theta = \pi$, when $r = \frac{l}{1-e}$. [Since $e < 1$, this value of r is positive.]



The curve therefore cuts the axis again at some point A' such that $SA' = \frac{l}{1-e}$.

As θ passes from π to 2π, $\cos\theta$ increases continuously from -1 to 1; hence $\frac{l}{r}$ increases continuously, and r decreases continuously from $\frac{l}{1-e}$ to $\frac{l}{1+e}$.

Since, for any value of θ, $\cos\theta = \cos(2\pi - \theta)$, the curve is symmetrical about the axis.

Therefore when e is less than unity, the equation represents a closed curve, symmetrical about the initial line.

(3) Let e be greater than unity, then the curve is an hyperbola.

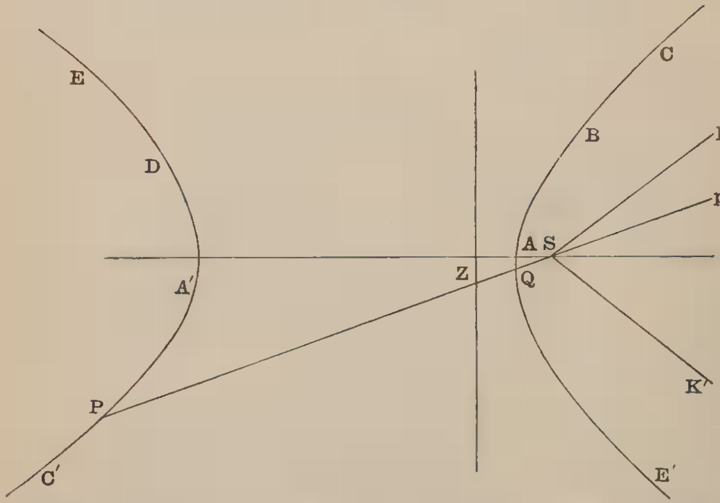
At the point A, $\theta = 0$ and $r = \frac{l}{1+e}$.

As θ increases $\cos\theta$ decreases, and therefore r increases until $1+e\cos\theta=0$. For this value of θ, which we will call α (the angle ASK in the figure), the value of r will be infinitely great.

As θ increases beyond the value α, $(1+e\cos\theta)$ becomes negative, and when $\theta=\pi$, $r=-\frac{l}{e-1}=SA'$ in the figure. $(1+e\cos\theta)$ will remain negative until θ is equal to $(2\pi-\alpha)$, the angle ASK' in the figure. When θ is equal to $(2\pi-\alpha)$, r is again infinite. If θ is somewhat less than this, r is very great and is *negative*, and if θ is somewhat greater, r is very great and is *positive*. The values of r will remain positive while θ changes from $(2\pi-\alpha)$ to 2π.

The curve is therefore described in the following order.

First the part ABC, then $C'PA'$ and $A'DE$, and lastly $E'QA$.



The curve consists of two separate branches, and the radius vector is negative for the whole of the branch $C'PA'DE$.

If, as in the figure, a line SQP be drawn cutting the curve in the two points Q and P which are on different branches, the two points Q and P must not be considered to have the same vectorial angle. The radius vector SP is *negative*, that is to say SP is drawn in the direction *opposite* to that which bounds its vectorial angle, the vectorial angle must therefore be ASp, p being on PS produced. So that, if the vectorial angle of Q be θ, that of P will be $\theta - \pi$.

163. *To find the polar equation of the straight line through two given points on a conic, and to find the equation of the tangent at any point.*

Let the vectorial angles of the two points P, Q be $(\alpha - \beta)$ and $(\alpha + \beta)$ respectively.

Let the equation of the conic be

$$\frac{l}{r} = 1 + e\cos\theta \quad\text{.................(i).}$$

The straight line whose equation is

$$\frac{l}{r} = A\cos\theta + B\cos(\theta - \alpha) \quad\text{...........(ii),}$$

will pass through any two points, since its equation contains the two independent constants A and B.

It will pass through the two points P, Q if r has the same values in (ii) as in (i) when $\theta = \alpha - \beta$, and when $\theta = \alpha + \beta$.

This will be the case, if

$$1 + e\cos(\alpha - \beta) = A\cos(\alpha - \beta) + B\cos\beta,$$

and

$$1 + e\cos(\alpha + \beta) = A\cos(\alpha + \beta) + B\cos\beta;$$

$$\therefore\ A = e, \text{ and } B\cos\beta = 1.$$

Substituting these values of A and B in (ii) we have the required equation of the chord, viz.

$$\frac{l}{r} = e\cos\theta + \sec\beta\cos(\theta - \alpha) \quad\text{.........(iii).}$$

To find the equation of the tangent at the point whose vectorial angle is α, we must put $\beta = 0$ in (iii), and we obtain

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) \quad\text{...........(iv).}$$

COR. If the equation of the conic be

$$\frac{l}{r} = 1 + e\cos(\theta - \gamma),$$

the chord joining the points $(\alpha - \beta)$ and $(\alpha + \beta)$ has for equation

$$\frac{l}{r} = e\cos(\theta - \gamma) + \sec\beta\cos(\theta - \alpha),$$

and the tangent at α has for equation

$$\frac{l}{r} = e\cos(\theta - \gamma) + \cos(\theta - \alpha).$$

164. *To find the equation of the polar of a point with respect to a conic.*

Let the equation of the conic be

$$\frac{l}{r} = 1 + e\cos\theta \quad \text{...................(i)},$$

and let the co-ordinates of the point be r_1, θ_1.

Let $\alpha \pm \beta$ be the vectorial angles of the points the tangents at which pass through (r_1, θ_1).

The equation of the line through these points will be

$$\frac{l}{r} = e\cos\theta + \sec\beta\cos(\theta - \alpha) \quad \text{.........(ii)}.$$

The equations of the tangents will be

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha + \beta),$$

and $$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha - \beta).$$

Since these pass through (r_1, θ_1), we have

$$\frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \alpha + \beta);$$

and $$\frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \alpha - \beta);$$

whence $$\theta_1 = \alpha, \text{ and } \cos\beta = \frac{l}{r_1} - e\cos\theta_1.$$

Substitute for α and β in (ii), and we have

$$\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)=\cos(\theta-\theta_1)\ldots\text{(iii)},$$

which is the required equation.

165. We will now solve some examples.

(1) The equation of the tangents at two points whose vectorial angles are α, β respectively are

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\alpha),$$

and

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\beta).$$

Where these meet

$$\cos(\theta-\alpha)=\cos(\theta-\beta);$$

$$\therefore \theta=\frac{\alpha+\beta}{2}.$$

Hence, *if T be the point of intersection of the tangents at the two points P, Q of a conic, ST will bisect the angle PSQ.* If however the conic be an hyperbola, and the points be on different branches of the curve, ST will bisect the *exterior* angle PSQ; for, as we have seen, the vectorial angle of P (if P be on the further branch) is not the angle which SP makes with SZ, but the angle PS produced makes with SZ.

(2) *If the tangent at any point P of a conic meet the directrix in K, then the angle KSP is a right angle.*

If the vectorial angle of P be α, the equation of the tangent at P will be

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\alpha).$$

This will meet the directrix, whose equation is $l=er\cos\theta$, where $\cos(\theta-\alpha)=0$.

Hence, at the point K, $\theta-\alpha=\pm\frac{\pi}{2}$.

Therefore the angle KSP is a right angle.

(3) *If chords of a conic subtend a constant angle at a focus, the tangents at the ends of the chord will meet on a fixed conic, and the chord will touch another fixed conic.*

Let 2β be the angle the chord subtends at the focus. Let $\alpha-\beta$ and $\alpha+\beta$ be the vectorial angles of the extremities of the chord.

The equation of the chord will be

$$\frac{l}{r} = e \cos\theta + \sec\beta \cos(\theta - \alpha),$$

or

$$\frac{l\cos\beta}{r} = e\cos\beta \,.\, \cos\theta + \cos(\theta - \alpha) \text{.............(i).}$$

But (i) is the equation of the tangent, at the point whose vectorial angle is α, to the conic whose equation is

$$\frac{l\cos\beta}{r} = 1 + e\cos\beta \,.\, \cos\theta \text{....................(ii).}$$

Hence the chord always touches a fixed conic, whose eccentricity is $e\cos\beta$, and semi-latus rectum $l\cos\beta$.

The equations of the tangents at the ends of the chord will be

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha + \beta),$$

and

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha - \beta).$$

Both these lines meet the conic

$$\frac{l}{r} = e\cos\theta + \cos\beta$$

in the same point, viz. where $\theta = \alpha$ and $\frac{l}{r} = e\cos\alpha + \cos\beta$.

Hence, the locus of the intersection of the tangents at the ends of the chord is the conic

$$\frac{l\sec\beta}{r} = 1 + e\sec\beta \,.\, \cos\theta \text{...................(iii).}$$

Both the conics (ii) and (iii) have the same focus and directrix as the given conic.

(4) *To find the equation of the circle circumscribing the triangle formed by three tangents to a parabola.*

Let the vectorial angles of the three points A, B, C be α, β, γ respectively.

Let the equation of the parabola be

$$\frac{l}{r} = 1 + \cos\theta.$$

The equations of the tangents at A, B, C respectively will be

$$\frac{l}{r} = \cos\theta + \cos(\theta - \alpha),$$

$$\frac{l}{r} = \cos\theta + \cos(\theta - \beta),$$

$$\frac{l}{r} = \cos\theta + \cos(\theta - \gamma).$$

The tangents at B and C meet where

$$\theta = \tfrac{1}{2}(\beta+\gamma), \text{ and } \therefore \frac{l}{r} = 2\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

The tangents at C and A meet where

$$\theta = \tfrac{1}{2}(\gamma+\alpha), \text{ and } \frac{l}{r} = 2\cos\frac{\gamma}{2}\cos\frac{\alpha}{2}.$$

And the tangents at A and B meet where

$$\theta = \tfrac{1}{2}(\alpha+\beta), \text{ and } \frac{l}{r} = 2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}.$$

By substitution we see that the three points of intersection are on the circle whose equation is

$$r = \frac{l}{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}\cos\left(\theta - \frac{\alpha}{2} - \frac{\beta}{2} - \frac{\gamma}{2}\right).$$

The circle always passes through the focus of the parabola.

(5) *To find the polar equation of the normal at any point of a conic, the focus being the pole.*

Let the equation of the conic be

$$\frac{l}{r} = 1 + e\cos\theta.$$

The equation of the tangent at any point α is

$$\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha).$$

The equation of any line perpendicular to the tangent is

$$\frac{C}{r} = e\cos\left(\theta+\frac{\pi}{2}\right) + \cos\left(\theta+\frac{\pi}{2}-\alpha\right),$$

or

$$\frac{C}{r} = -e\sin\theta - \sin(\theta-\alpha).$$

This will be the required equation of the normal provided C is so chosen that the point $\left(\frac{l}{1+e\cos\alpha}, \alpha\right)$ may be on the line. Hence we must have

$$C\frac{1+e\cos\alpha}{l} = -e\sin\alpha,$$

or

$$C = \frac{-le\sin\alpha}{1+e\cos\alpha}.$$

Hence the equation of the normal is

$$\frac{le\sin\alpha}{1+e\cos\alpha}\cdot\frac{1}{r} = e\sin\theta + \sin(\theta-\alpha).$$

EXAMPLES ON CHAPTER VIII.

1. The exterior angle between any two tangents to a parabola is equal to half the difference of the vectorial angles of their points of contact.

2. The locus of the point of intersection of two tangents to a parabola which cut one another at a constant angle is a hyperbola having the same focus and directrix as the original parabola.

3. If PSP' and QSQ' be any two focal chords of a conic at right angles to one another, shew that $\frac{1}{PS\,.\,SP'}+\frac{1}{QS\,.\,SQ'}$ is constant.

4. If A, B, C be any three points on a parabola, and the tangents at these points form a triangle $A'B'C'$, shew that $SA\,.\,SB\,.\,SC = SA'\,.\,SB'\,.\,SC'$, S being the focus of the parabola.

5. If a focal chord of an ellipse make an angle α with the axis, the angle between the tangents at its extremities is

$$\tan^{-1}\frac{2e\sin\alpha}{1-e^2}.$$

6. By means of the equation $\frac{l}{r}=1+e\cos\theta$, shew that the ellipse might be generated by the motion of a point moving so that the sum of its distances from two fixed points is constant.

7. Find the locus of the pole of a chord which subtends a constant angle (2α) at a focus of a conic, distinguishing the cases for which $\cos\alpha >=< e$.

8. PQ is a chord of a conic which subtends a right angle at a focus. Shew that the locus of the pole of PQ and the locus enveloped by PQ are each conics whose latera recta are to that of the original conic as $\sqrt{2}:1$ and $1:\sqrt{2}$ respectively.

9. Given the focus and directrix of a conic, shew that the polar of a given point with respect to it passes through a fixed point.

10. If two conics have a common focus, shew that two of their common chords will pass through the point of intersection of their directrices.

11. Two conics have a common focus and any chord is drawn through the focus meeting the conics in P, P' and Q, Q' respectively. Shew that the tangents at P or P' meet those at Q, Q' in points lying on two straight lines through the intersection of the directrices, these lines being at right angles if the conics have the same eccentricity.

12. Through the focus of a parabola any two chords LSL', MSM' are drawn; the tangent at L meets those at M, M' in the points N, N' and the tangent at L' meets them in K', K. Shew that the lines KN, $K'N'$ are at right angles.

13. Two conics have a common focus about which one is turned; shew that two of their common chords will touch conics having the fixed focus for focus.

14. Shew that the equation of the locus of the point of intersection of two tangents to $\frac{l}{r} = 1 + e\cos\theta$, which are at right angles to one another, is $r^2(e^2 - 1) - 2le\,r\cos\theta + 2l^2 = 0$.

15. If PSQ, PHR be two chords of an ellipse through the foci S, H, then will $\frac{PS}{SQ} + \frac{PH}{HR}$ be independent of the position of P.

16. Two conics are described having the same focus, and the distance of this focus from the corresponding directrix of each is the same; if the conics touch one another, prove that twice the sine of half the angle between the transverse axes is equal to the difference of the reciprocals of the eccentricities.

17. A circle of given radius passing through the focus of a given conic intersects it in A, B, C, D; shew that

$$SA \,.\, SB \,.\, SC \,.\, SD$$

is constant.

18. A circle passing through the focus of a conic whose latus rectum is $2l$ meets the conic in four points whose distances from the focus are r_1, r_2, r_3, r_4; prove that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{l}$.

19. A given circle whose centre is on the axis of a parabola passes through the focus S, and is cut in four points A, B, C, D by any conic of given latus rectum having S for focus and a tangent to the parabola for directrix; shew that the sum of the distances SA, SB, SC, SD is constant.

20. Two points P, Q are taken one on each of two conics, which have a common focus and their axes in the same direction, such that PS and QS are at right angles, S being the common focus. Shew that the tangents at P and Q meet on a conic the square of whose eccentricity is equal to the sum of the squares of the eccentricities of the original conics.

21. A series of conics are described with a common latus rectum; prove that the locus of points upon them, at which the perpendicular from the focus on the tangent is equal to the semi-latus rectum, is given by the equation $l = -r\cos 2\theta$.

22. If POP' be a chord of a conic through a fixed point O, then will $\tan \frac{1}{2} P'SO \tan \frac{1}{2} PSO$ be constant, S being a focus of the conic.

23. Conics are described with equal latera recta and a common focus. Also the corresponding directrices envelope a fixed confocal conic. Prove that these conics all touch two fixed conics, the reciprocals of whose latera recta are the sum and difference respectively of those of the variable conic and their fixed confocal and which have the same directrix as the fixed confocal.

Review rectangular hyp
p 17, 48-9, 50

CHAPTER IX.

GENERAL EQUATION OF THE SECOND DEGREE.

166. We have seen in the preceding Chapters that the equation of a conic is always of the second degree: we shall now prove that every equation of the second degree represents a conic, and shew how to determine from any such equation the nature and position of the conic which it represents.

167. *To shew that every curve whose equation is of the second degree is a conic.*

We may suppose the axes of co-ordinates to be rectangular; for if the equation be referred to oblique axes, and we change to rectangular axes, the degree of the equation is not altered [Art. 53].

Let then the equation of the curve be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\text{.........(i)}.$$

As this is the most general form of the equation of the second degree it will include all possible cases.

We can get rid of the term containing xy by turning the axes through a certain angle.

For, to turn the axes through an angle θ we have to substitute for x and y respectively $x\cos\theta - y\sin\theta$, and $x\sin\theta + y\cos\theta$ [Art. 50].

The equation (i) will become

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + b(x\sin\theta + y\cos\theta)^2 + 2g(x\cos\theta - y\sin\theta) + 2f(x\sin\theta + y\cos\theta) + c = 0 \quad \ldots\ldots (ii).$$

The coefficient of xy in (ii) is

$$2(b-a)\sin\theta\cos\theta + 2h(\cos^2\theta - \sin^2\theta);$$

and this will be zero, if

$$\tan 2\theta = \frac{2h}{a-b} \quad \ldots\ldots (iii).$$

Since an angle can be found whose tangent is equal to any real quantity whatever, the angle $\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$ is in all cases real.

Equation (ii) may now be written

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0 \ldots\ldots (iv).$$

If neither A nor B be zero, we can write equation (iv) in the form

$$A\left(x + \frac{G}{A}\right)^2 + B\left(y + \frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - C$$

or, taking the origin at the point $\left(-\frac{G}{A}, -\frac{F}{B}\right)$,

$$Ax^2 + By^2 = \frac{G^2}{A} + \frac{F^2}{B} - C \ldots\ldots (v).$$

If the right side of (v) be zero, the equation will represent *two straight lines* [Art. 35].

If however the right side of (v) be not zero, we have the equation

$$\frac{x^2}{\frac{1}{A}\left(\frac{G^2}{A} + \frac{F^2}{B} - C\right)} + \frac{y^2}{\frac{1}{B}\left(\frac{G^2}{A} + \frac{F^2}{B} - C\right)} = 1,$$

which we know represents an *ellipse* if both denominators are positive, and an *hyperbola* if one denominator is positive and the other negative.

If both denominators are negative, it is clear that no real values of x and of y will satisfy the equation. In this case the curve is an *imaginary ellipse.*

Next let A or B be zero, A suppose. [A and B cannot *both* be zero by Art. 53.] Equation (iv) can then be written

$$B\left(y+\frac{F}{B}\right)^2=-2Gx-C+\frac{F^2}{B}\text{ ....(vi).}$$

If $G=0$, this equation represents a pair of *parallel straight lines.*

If G be not zero, we may write the equation

$$\left(y+\frac{F}{B}\right)^2=-\frac{2G}{B}\left(x-\frac{F^2}{2BG}+\frac{C}{2G}\right),$$

which represents a *parabola,* whose axis is parallel to the axis of x.

Hence in all cases the curve represented by the general equation of the second degree is a conic.

168. *To find the co-ordinates of the centre of a conic.*

We have seen [Art. 109] that when the origin of coordinates is the centre of a conic its equation does not contain any terms involving the first power of the variables. To find the centre of the conic, we must therefore change the origin to some point (x', y'), and choose x', y', so that the coefficients of x and y in the transformed equation may be zero.

Let the equation of the conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

The equation referred to parallel axes through the point (x', y') will be found by substituting $x+x'$ for x, and $y+y'$ for y, and will therefore be

$$a(x+x')^2+2h(x+x')(y+y')+b(y+y')^2+2g(x+x') \\ +2f(y+y')+c=0,$$

or $$ax^2+2hxy+by^2+2x(ax'+hy'+g)+2y(hx'+by'+f) \\ +ax'^2+2hx'y'+by'^2+2gx'+2fy'+c=0.$$

The coefficients of x and y will both be zero in the above, if x' and y' be so chosen that

$$ax'+hy'+g=0\text{ ....(i),}$$

and $$hx'+by'+f=0\text{ ....(ii).}$$

The equation referred to (x', y') as origin will then be

$$ax^2+2hxy+by^2+c'=0\text{ ....(iii),}$$

where $c' = ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c$......(iv). Hence the co-ordinates of the centre of the conic are the values of x' and y' given by the equations (i) and (ii).

The centre is therefore the point

$$\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right).$$

When $ab - h^2 = 0$, the co-ordinates of the centre are infinite, and the curve is therefore a parabola [Art. 157].

If however $hf - bg = 0$ and $ab - h^2 = 0$; that is, if

$$\frac{a}{h} = \frac{h}{b} = \frac{g}{f},$$

the equations (i) and (ii) represent the same straight line, and any point of that line is a centre. The locus in this case is a pair of parallel straight lines.

In the above investigation the axes may be either rectangular or oblique.

Subsequent investigations which hold good for oblique axes will be distinguished by the sign (ω).

169. Multiply equations (i) and (ii) of the preceding Article by x', y' respectively, and subtract the sum from the right-hand member of (iv); then we have

$$\begin{aligned} c' &= gx' + fy' + c \\ &= g\frac{hf - bg}{ab - h^2} + f\frac{gh - af}{ab - h^2} + c \\ &= \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2}. \qquad (\omega). \end{aligned}$$

170. The expression $abc + 2fgh - af^2 - bg^2 - ch^2$ is usually denoted by the symbol Δ, and is called the *discriminant* of

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c.$$

$\Delta = 0$ is the condition that the conic may be two straight lines.

For, if Δ is zero, c' is zero; and in that case equation (iii) Art. 168 will represent two straight lines.

This is the condition we found in Art. 37. (ω).

171. *To find the position and magnitude of the axes of the conic whose equation is* $ax^2 + 2hxy + by^2 = 1$.

If a conic be cut by any concentric circle, the diameters through the points of intersection will be equally inclined to the axes of the conic, and will be coincident if the radius of the circle be equal to either of the semi-axes of the conic.

Now the lines through the origin and through the points of intersection of the conic and the circle whose equation is $x^2 + y^2 = r^2$, are given by the equation

$$\left(a - \frac{1}{r^2}\right) x^2 + 2hxy + \left(b - \frac{1}{r^2}\right) y^2 = 0 \ldots\ldots\text{(i)}.$$

These lines will be *coincident*, if

$$\left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) - h^2 = 0 \ldots\ldots\ldots\ldots\text{(ii)},$$

and they will then coincide with one or other of the axes of the conic.

Hence the lengths of the semi-axes of the conic are the roots of the equation (ii), that is of the equation

$$\frac{1}{r^4} - (a + b)\frac{1}{r^2} + ab - h^2 = 0 \ldots\ldots\ldots\text{(iii)}.$$

Multiply (i) by $\left(a - \frac{1}{r^2}\right)$; then, if $\frac{1}{r^2}$ is either of the roots of the equation (ii), we get

$$\left(a - \frac{1}{r^2}\right)^2 x^2 + 2h\left(a - \frac{1}{r^2}\right) xy + h^2y^2 = 0;$$

whence $$\left(a - \frac{1}{r^2}\right) x + hy = 0 \ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

Hence if we substitute in (iv) either root of the equation (iii) we get the equation of the corresponding axis.

In the above we have supposed the axes to be rectangular. If however they are inclined at an angle ω the investigation must be slightly modified, for the equation of the circle of radius r will be $x^2 + 2xy\cos\omega + y^2 = r^2$.

172. *To find the axis and latus rectum of a parabola.*

If the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represent a parabola, the terms of the second degree form a perfect square. [This follows from the fact that the equation of any parabola can be expressed in the form $y^2 - 4a'x = 0$, and therefore with any axes the equation will be of the form

$$(lx + my + n)^2 - 4a'(l'x + m'y + n') = 0.]$$

Hence the equation is equivalent to

$$(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots \text{(i)},$$

where $\alpha^2 = a$, and $\beta^2 = b$.

From (i) we see that the square of the perpendicular on the line $\alpha x + \beta y = 0$ varies as the perpendicular on the line $2gx + 2fy + c = 0$. These lines may not be at right angles, but we may write the equation (i) in the form

$$(\alpha x + \beta y + \lambda)^2 = 2x(\lambda\alpha - g) + 2y(\lambda\beta - f) + \lambda^2 - c,$$

and the two straight lines, whose equations are

$\alpha x + \beta y + \lambda = 0$, and $2x(\lambda\alpha - g) + 2y(\lambda\beta - f) + \lambda^2 - c = 0$,

will be at right angles to one another, if

$$\alpha(\lambda\alpha - g) + \beta(\lambda\beta - f) = 0,$$

or if

$$\lambda = \frac{\alpha g + \beta f}{\alpha^2 + \beta^2}.$$

Now take

$\alpha x + \beta y + \lambda = 0$ and $2(\alpha\lambda - g)x + 2(\beta\lambda - f)y + \lambda^2 - c = 0$

for new axes of x and y respectively, and we get

$$y^2 = 4px,$$

and this we know is the equation of a parabola referred to its axis and the tangent at the vertex.

To find the latus-rectum, we write the equation in the form

$$\left(\frac{\alpha x + \beta y + \lambda}{\sqrt{\alpha^2 + \beta^2}}\right)^2 = 4p\left\{\frac{2(\alpha\lambda - g)x + 2(\beta\lambda - f)y + \lambda^2 - c}{\sqrt{\{4(\alpha\lambda - g)^2 + 4(\beta\lambda - f)^2\}}}\right\};$$

hence $$4p = \frac{\sqrt{\{4(\alpha\lambda - g)^2 + 4(\beta\lambda - f)^2\}}}{\alpha^2 + \beta^2}.$$

Hence (i) is a parabola whose axis is the line

$$\alpha x + \beta y + \lambda = 0,$$

and whose latus-rectum is

$$\frac{2\sqrt{\{(\alpha\lambda - g)^2 + (\beta\lambda - f)^2\}}}{\alpha^2 + \beta^2} = \frac{2(\alpha f - \beta g)}{(\alpha^2 + \beta^2)^{\frac{3}{2}}},$$

since $$\lambda = \frac{\alpha g + \beta f}{\alpha^2 + \beta^2}.$$

173. We will now find the nature and position of the conics given by the following equations.

(1) $7x^2 - 17xy + 6y^2 + 23x - 2y - 20 = 0$.

(2) $x^2 - 5xy + y^2 + 8x - 20y + 15 = 0$.

(3) $36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0$.

(4) $(5x - 12y)^2 - 2x - 29y - 1 = 0$.

(1) The equations for finding the centre are [Art. 168, (i), (ii)]

$$\left.\begin{array}{r} 14x' - 17y' + 23 = 0 \\ -17x' + 12y' - 2 = 0 \end{array}\right\}.$$

These give $x' = 2$, $y' = 3$. Therefore centre is the point (2, 3).

The equation referred to parallel axes through the centre will be [Art. 169]

$$7x^2 - 17xy + 6y^2 + \frac{23}{2}.2 - 1.3 - 20 = 0.$$

or $$7x^2 - 17xy + 6y^2 = 0.$$

The equation therefore represents two straight lines which intersect in the point (2, 3). They cut the axis of x, where $7x^2 + 23x - 20 = 0$, that is where $x = -4$, and where $x = \frac{5}{7}$.

(2) $x^2 - 5xy + y^2 + 8x - 20y + 15 = 0$.

The equations for finding the centre are

$$2x' - 5y' + 8 = 0, \text{ and } -5x' + 2y' - 20 = 0;$$
$$\therefore x' = -4, \ y' = 0.$$

The equation referred to parallel axes through the centre will be

$$x^2 - 5xy + y^2 + 4(-4) + 15 = 0,$$

or $$x^2 - 5xy + y^2 = 1,$$

The semi-axes of the conic are the roots of the equation

$$\frac{1}{r^4}-(a+b)\frac{1}{r^2}+ab-h^2=0 \quad [\text{Art. 171, (iii)}];$$

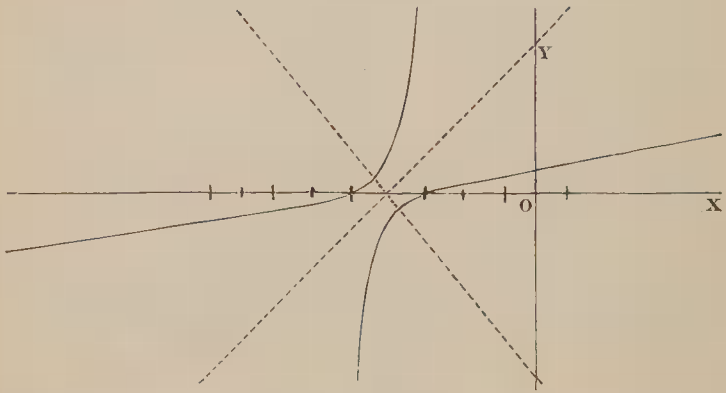
$$\therefore \frac{1}{r^4}-\frac{2}{r^2}+1-\frac{25}{4}=0,$$

or

$$21r^4+8r^2-4=0;$$

$$\therefore r^2=\frac{2}{7}, \text{ or } -\frac{2}{3}.$$

The curve is therefore an hyperbola whose real semi-axis is $\frac{1}{7}\sqrt{14}$, and whose imaginary semi-axis is $\frac{1}{3}\sqrt{-6}$.



The direction of the real axis is given [Art. 171, (iv)] by the equation

$$\left(1-\frac{7}{2}\right)x-\frac{5}{2}y=0,$$

or

$$x+y=0.$$

(3) $36x^2+24xy+29y^2-72x+126y+81=0.$

The equations for finding the centre are

$$36x'+12y'-36=0, \text{ and } 12x'+29y'+63=0;$$

$$\therefore x'=2, \; y'=-3.$$

The equation referred to parallel axes through the centre, will be

$$36x^2+24xy+29y^2-72+63(-3)+81=0,$$

or

$$\frac{x^2}{5}+\frac{2}{15}xy+\frac{29}{180}y^2=1.$$

The semi-axes of the conic are the roots of the equation

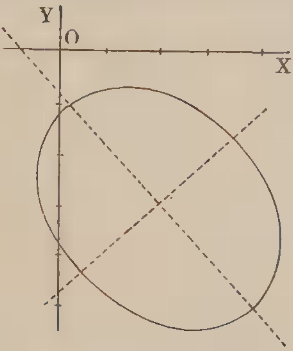
$$\frac{1}{r^4}-(a+b)\frac{1}{r^2}+ab-h^2=0.$$

And
$$a+b=\frac{65}{180}=\frac{13}{36},$$

$$ab-h^2=\frac{29}{900}-\frac{1}{225}=\frac{1}{36};$$

$$\therefore\ 36-13r^2+r^4=0.$$

Hence the squares of the semi-axes are 9 and 4.



The equation of the major axis is [Art. 171, (iv)]

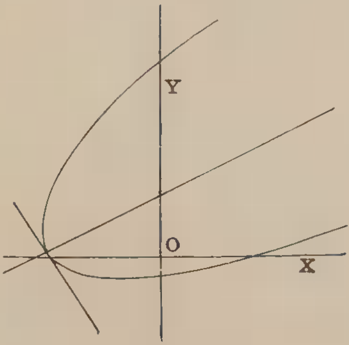
$$\left(\frac{1}{5}-\frac{1}{9}\right)x+\frac{1}{15}y=0,$$

or
$$4x+3y=0.$$

(4) $(5x-12y)^2-2x-29y-1=0.$

The equation may be written

$$(5x-12y+\lambda)^2=2x(1+5\lambda)+y(29-24\lambda)+\lambda^2+1.$$



The lines $$5x-12y+\lambda=0$$
and $$2(1+5\lambda)x+(29-24\lambda)y+\lambda^2+1=0$$
are at right angles, if
$$10+50\lambda-348+288\lambda=0;$$
that is, if $\lambda=1$.

The equation is therefore equivalent to
$$\left(\frac{5x-12y+1}{13}\right)^2=\frac{1}{13}\cdot\frac{12x+5y+2}{13}\quad\ldots\ldots\ldots\ldots\ldots\text{(i)};$$
therefore $5x-12y+1=0$ is the equation of the axis of the parabola, and $12x+5y+2=0$ is the equation of the tangent at the vertex.

Every point on the curve must clearly be on the positive side of the line $12x+5y+2=0$, since the left side of equation (i) is always positive.

174. *To find the equation of the asymptotes of a conic.*

We have seen [Art. 146] that the equations of a conic and of the asymptotes only differ by a constant.

Let the equation of a conic be
$$ax^2+2hxy+by^2+2gx+2fy+c=0\ \ldots\ldots\text{(i)}.$$

Then the equations of the asymptotes will be
$$ax^2+2hxy+by^2+2gx+2fy+c+\lambda=0\ldots\ldots\text{(ii)},$$
provided we give to λ that value which will make (ii) represent a pair of straight lines.

The condition that (ii) may represent a pair of straight lines is [Art. 170]
$$ab(c+\lambda)+2fgh-af^2-bg^2-(c+\lambda)h^2=0;$$
$$\therefore\ \lambda(ab-h^2)+\Delta=0.$$

Hence the equation of the asymptotes of (i) is
$$ax^2+2hxy+by^2+2gx+2fy+c-\frac{\Delta}{ab-h^2}=0.$$

The equations of two conjugate hyperbolas differ from the equation of their asymptotes by constants which are equal and opposite to one another [Art. 152]; therefore the equation of the hyperbola conjugate to (i) is
$$ax^2+2hxy+by^2+2gx+2fy+c-\frac{2\Delta}{ab-h^2}=0.$$

Cor. The lines represented by the equation

$$ax^2 + 2hxy + by^2 = 0$$

are parallel to the asymptotes of the conic. (ω).

Ex. Find the asymptotes of the conic

$$x^2 - xy - 2y^2 + 3y - 2 = 0.$$

The asymptotes will be $x^2 - xy - 2y^2 + 3y - 2 + \lambda = 0$, if this equation represents straight lines. Solving as a quadratic in x, we have

$$x = \frac{y}{2} \pm \sqrt{\left\{\frac{9}{4}y^2 - 3y + 2 - \lambda\right\}}.$$

Hence [Art. 37], the condition for straight lines is $9(2 - \lambda) = 9$, or $\lambda = 1$. The asymptotes are therefore $x^2 - xy - 2y^2 + 3y - 1 = 0$.

175. *To find the condition that the conic represented by the general equation of the second degree may be a rectangular hyperbola.*

If the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

the equation

$$ax^2 + 2hxy + by^2 = 0 \text{ ....(i)}$$

represents straight lines parallel to the asymptotes.

Hence, if the conic is a rectangular hyperbola, the lines given by (i) must be at right angles.

The required condition is therefore [Art. 44]

$$a + b - 2h\cos\omega = 0 \text{ ....(ii).}$$

If the axes of co-ordinates be at right angles to one another the condition is

$$a + b = 0 \text{...................(iii).}$$

The required condition may also be found as follows. If the axes of co-ordinates be changed in any manner whatever, we have

$$\frac{a + b - 2h\cos\omega}{\sin^2\omega} = \frac{a' + b' - 2h'\cos\omega'}{\sin^2\omega'} \text{ [Art. 52].}$$

But, if the conic be a rectangular hyperbola and the asymptotes be taken for axes, the equation will be $xy + \text{constant} = 0$;

$$\therefore \quad a' = b' = \cos\omega' = 0.$$

Hence $$a + b - 2h\cos\omega = 0. \qquad (\omega).$$

EXAMPLES ON CHAPTER IX.

1. Find the centres of the following curves:

(i) $3x^2 - 5xy + 6y^2 + 11x - 17y + 13 = 0$.
(ii) $xy + 3ax - 3ay = 0$.
(iii) $3x^2 - 7xy - 6y^2 + 3x - 9y + 5 = 0$.

Find also the equations of the curves referred to parallel axes through their centres.

2. What do the following equations represent?

(i) $xy - 2x + y - 2 = 0$. (ii) $y^2 - 2ay + 4ax = 0$.
(iii) $y^2 + ax + ay + a^2 = 0$. (iv) $(x + y)^2 = a(x - y)$.
(v) $4(x + 2y)^2 + (y - 2x)^2 = 5a^2$. (vi) $y^2 - x^2 - 2ax = 0$.

3. Draw the following curves:

(1) $xy + ax - 2ay = 0$. (2) $x^2 + 2xy + y^2 - 2x - 1 = 0$.
(3) $2x^2 + 5xy + 2y^2 + 3y - 2 = 0$.
(4) $x^2 + 4xy + y^2 - 11 = 0$.
(5) $(2x + 3y)^2 + 2x + 2y + 2 = 0$.
(6) $x^2 - 4xy - 2y^2 + 10x + 4y = 0$.
(7) $41x^2 + 24xy + 9y^2 - 130ax - 60ay + 116a^2 = 0$.

4. Shew that if two chords of a conic bisect each other, their point of intersection must be the centre of the curve.

5. Shew that the product of the semi-axes of the conic whose equation is

$$(x - 2y + 1)^2 + (4x + 2y - 3)^2 - 10 = 0, \text{ is } 1.$$

6. Shew that the product of the semi-axes of the ellipse whose equation is

$$x^2 - xy + 2y^2 - 2x - 6y + 7 = 0 \text{ is } \frac{2}{\sqrt{7}};$$

and that the equation of its axes is

$$x^2 - y^2 - 2xy + 8y - 8 = 0.$$

7. Find for what value of λ the equation

$$2x^2 + \lambda xy - y^2 - 3x + 6y - 9 = 0$$

will represent a pair of straight lines.

8. Find the equation of the conic whose asymptotes are the lines $2x + 3y - 5 = 0$ and $5x + 3y - 8 = 0$, and which passes through the point $(1, -1)$.

9. Find the equation of the asymptotes of the conic

$$3x^2 - 2xy - 5y^2 + 7x - 9y = 0;$$

and find the equation of the conic which has the same asymptotes and which passes through the point $(2, 2)$.

10. Find the asymptotes of the hyperbola

$$6x^2 - 7xy - 3y^2 - 2x - 8y - 6 = 0;$$

find also the equation of the conjugate hyperbola.

11. Shew that, if

$$ax^2 + 2hxy + by^2 = 1, \text{ and } a'x^2 + 2h'xy + b'y^2 = 1$$

represent the same conic, and the axes are rectangular, then

$$(a - b)^2 + 4h^2 = (a' - b')^2 + 4h'^2.$$

12. Shew that for all positions of the axes so long as they remain rectangular, and the origin is unchanged, the value of $g^2 + f^2$ in the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is constant.

13. From any point on a given straight line tangents are drawn to each of two circles: shew that the locus of the point of intersection of the chords of contact is a hyperbola whose asymptotes are perpendicular to the given line and to the line joining the centres of the two circles.

14. A variable circle always passes through a fixed point O and cuts a conic in the points P, Q, R, S; shew that

$$\frac{OP \,.\, OQ \,.\, OR \,.\, OS}{(\text{radius of circle})^2}$$

is constant.

15. If $ax^2 + 2hxy + by^2 = 1$, and $Ax^2 + 2Hxy + By^2 = 1$ be the equations of two conics, then will $aA + bB + 2hH$ be unaltered by any change of rectangular axes.

CHAPTER X.

MISCELLANEOUS PROPOSITIONS.

176. WE have proved [Art. 167] that the curve represented by an equation of the second degree is always a conic.

We shall throughout the present chapter assume that the equation of the conic is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

unless it is otherwise expressed.

The left-hand side of this equation will be sometimes denoted by $\phi(x, y)$.

177. *To find the equation of the straight line passing through two points on a conic, and to find the equation of the tangent at any point.*

Let (x', y') and (x'', y'') be two points on the conic.

The equation

$$a(x-x')(x-x'') + h\{(x-x')(y-y'') + (x-x'')(y-y')\}$$
$$+ b(y-y')(y-y'') = ax^2 + 2hxy + by^2 + 2gx + 2fy + c \ \ldots \text{(i)}$$

when simplified is of the first degree, and therefore represents *some* straight line.

If we put $x = x'$ and $y = y'$ in (i) the left side vanishes identically, and the right side vanishes since (x', y') is on the conic. Hence the point (x', y') is on the line (i). So also the point (x'', y'') is on the line (i).

Hence the equation of the straight line through the two points (x', y') and (x'', y'') is (i), and this reduces to

$$ax(x' + x'') + hy(x' + x'') + hx(y' + y'') + by(y' + y'') + 2gx + 2fy + c = ax'x'' + h(x'y'' + y'x'') + by'y'' \ldots \text{(ii)}.$$

To obtain the tangent at (x', y') we put $x'' = x'$, and $y'' = y'$ in (ii), and we get

$$2axx' + 2h(xy' + x'y) + 2byy' + 2gx + 2fy + c = ax'^2 + 2hx'y' + by'^2.$$

Add $2gx' + 2fy' + c$ to both sides: then, since (x', y') is on the conic, the right side will vanish; and we get for the equation of the tangent

$$ax'x + h(y'x + x'y) + by'y + g(x + x') + f(y + y') + c = 0.$$

It should be noticed that the equation of the tangent at (x', y') is obtained from the equation of the curve by writing $x'x$ for x^2, $y'x + x'y$ for $2xy$, $y'y$ for y^2, $x + x'$ for $2x$, and $y + y'$ for $2y$. (ω).

178. *To find the condition that a given straight line may be a tangent to a conic.*

Let the equation of the straight line be

$$lx + my + n = 0 \ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

The equation of the straight lines joining the origin to the points where the line (i) cuts the conic $\phi(x, y) = 0$, are given [Art. 38] by the equation

$$ax^2 + 2hxy + by^2 - 2(gx + fy)\frac{lx + my}{n} + c\left(\frac{lx + my}{n}\right)^2 = 0 \ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

If the line (i) be a tangent it will cut the conic in coincident points, and therefore the lines (ii) must be coincident. The condition for this is

$$(an^2 - 2gln + cl^2)(bn^2 - 2fmn + cm^2)$$
$$= (hn^2 - fln - gmn + clm)^2,$$

or $$l^2(bc - f^2) + m^2(ca - g^2) + n^2(ab - h^2) + 2mn(gh - fa)$$
$$+ 2nl(hf - gb) + 2lm(fg - hc) = 0 \ldots\ldots\text{(iii)}.$$

The equation (iii) may be written in the form

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0 \ldots\text{(iv)},$$

where the coefficients A, B, C, &c. are the minors of a, b, c, &c. in the determinant

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \cdot \qquad (\omega).$$

179. *To find the equation of the polar of any point with respect to a conic.*

It may be shewn, exactly as in Article 76, 100, or 118, that the equation of the polar is of the *same form* as the equation of the tangent.

The equation of the polar of (x', y') is therefore

$$ax'x + h(y'x + x'y) + by'y + g(x + x') + f(y + y') + c = 0,$$

or $$x(ax' + hy' + g) + y(hx' + by' + f) + gx' + fy' + c = 0.$$

The equation of the polar of the origin is found by putting $x' = y' = 0$ in the above; the result is

$$gx + fy + c = 0.$$

180. *If two points P, Q be such that Q is on the polar of P with respect to a conic, then will P be on the polar of Q with respect to that conic.*

Let the co-ordinates of P be x', y', and those of Q x'', y''.

The equation of the polar of P is

$$ax'x + h(y'x + x'y) + by'y + g(x + x') + f(y + y') + c = 0.$$

Since (x'', y'') is on the polar of P, we have

$$ax'x'' + h(y'x'' + x'y'') + by'y'' + g(x' + x'') + f(y' + y'') + c = 0.$$

The symmetry of this result shews that it is also the condition that the polar of Q should pass through P.

If the polars of two points P, Q meet in R, then R is the pole of the line PQ.

For, since R is on the polar of P, the polar of R will go through P; similarly the polar of R will go through Q; and therefore it must be the line PQ.

If any chord of a conic be drawn through a fixed point Q, and P be the pole of the chord; then, since Q is on the polar of P, the point P will always lie on a fixed straight line, namely on the polar of Q.

Def. Two points are said to be *conjugate* with respect to a conic when each lies on the polar of the other.

Def. Two straight lines are said to be *conjugate* with respect to a conic when each passes through the pole of the other. Conjugate diameters, as defined in Art. 127, are conjugate lines through the centre.

181. *If any chord of a conic be drawn through a point O it will be cut harmonically by the curve and the polar of O.*

Let $OPQR$ be any chord which cuts a conic in P, R and the polar of O with respect to the conic in Q.

Take O for origin, and the line $OPQR$ for axis of x; and let the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Where $y = 0$ cuts the conic we have

$$ax^2 + 2gx + c = 0;$$

$$\therefore \frac{1}{OP} + \frac{1}{OR} = -\frac{2g}{c} \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

The equation of the polar of O is

$$gx + fy + c = 0;$$

$$\therefore \frac{1}{OQ} = -\frac{g}{c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

From (i) and (ii) we see that

$$\frac{1}{OP} + \frac{1}{OR} = \frac{2}{OQ}.$$

182. *To find the locus of the middle points of a system of parallel chords of a conic.*

Let (x', y') and (x'', y'') be two points on the conic. The equation

$$a(x-x')(x-x'')+h\{(x-x')(y-y'')+(x-x'')(y-y')\}$$
$$+b(y-y')(y-y'')=ax^2+2hxy+by^2+2gx+2fy+c \quad \text{...(i)}$$

is the equation of the straight line joining the two points.

In (i) the coefficient of x is $a(x'+x'')+h(y'+y'')+2g$, and the coefficient of y is $h(x'+x'')+b(y'+y'')+2f$; hence if the line is parallel to the line $y=mx$, we have

$$m=-\frac{a(x'+x'')+h(y'+y'')+2g}{h(x'+x'')+b(y'+y'')+2f} \quad \text{.......(ii).}$$

Now, if (x, y) be the middle point of the chord joining (x', y') and (x'', y''), then $2x=x'+x''$, and $2y=y'+y''$; therefore, from (ii), we have

$$m=-\frac{ax+hy+g}{hx+by+f},$$

or $$x(a+mh)+y(h+mb)+g+mf=0 \quad \text{...(iii),}$$

which is the required equation.

If the line (iii) be written in the form $y=m'x+k$, then we have

$$m'=-\frac{a+mh}{h+mb},$$

or $$a+h(m+m')+bmm'=0 \quad \text{.........(iv).}$$

This is the condition that the lines $y=mx$ and $y=m'x$ may be parallel to conjugate diameters of the conic given by the general equation of the second degree. (ω).

183. *To find the condition that the two lines given by the equation* $Ax^2+2Hxy+By^2=0$ *may be conjugate diameters of the conic* $ax^2+2hxy+by^2=1$.

If the lines given by the equation $Ax^2+2Hxy+By^2=0$ be the same as $y-mx=0$, and $y-m'x=0$; then

$$m+m'=-2\frac{H}{B}, \text{ and } mm'=\frac{A}{B}.$$

But $y-mx=0$ and $y-m'x=0$ are conjugate diameters if

$$a+h(m+m')+bmm'=0.$$

Therefore the required condition is

$$a-2h\,.\,\frac{H}{B}+b\,\frac{A}{B}=0,$$

or

$$aB+bA=2hH. \qquad (\omega).$$

[The above result follows at once from Articles 155 and 58.]

Ex. 1. *To find the equation of the equi-conjugate diameters of the conic*

$$ax^2+2hxy+by^2=1.$$

The straight lines through the centre of a conic and any concentric circle give *equal* diameters. Through the intersections of the conic and the circle whose equation is $\lambda(x^2+y^2+2xy\cos\omega)=1$, the lines

$$(a-\lambda)x^2+2(h-\lambda\cos\omega)xy+(b-\lambda)y^2=0 \text{ pass.}$$

These are conjugate if

$$b(a-\lambda)+a(b-\lambda)=2h(h-\lambda\cos\omega).$$

Substituting the value of λ so found, we have the required equation

$$ax^2+2hxy+by^2-\frac{2(ab-h^2)}{a+b-2h\cos\omega}(x^2+y^2+2xy\cos\omega)=0.$$

Ex. 2. *To shew that any two concentric conics have in general one and only one pair of common conjugate diameters.*

Let the equations of the two conics be

$$ax^2+2hxy+by^2=1, \text{ and } a'x^2+2h'xy+b'y^2=1.$$

The diameters $Ax^2+2Hxy+By^2=0$ are conjugate with respect to both conics if

$$Ab-2Hh+Ba=0,$$

and

$$Ab'-2Hh'+Ba'=0;$$

$$\therefore \frac{A}{ha'-ah'}=\frac{-2H}{ab'-a'b}=\frac{B}{bh'-b'h}.$$

The equation of the common conjugate diameters is therefore

$$(ha'-ah')x^2-(ab'-a'b)xy+(bh'-b'h)y^2=0.$$

Since any two concentric conics have one pair of conjugate diameters in common, it follows that the equations of any two concentric conics can be reduced to the forms

$$ax^2+by^2=1, \quad a'x^2+b'y^2=1.$$

184. *To find the length of a straight line drawn from a given point in a given direction to meet a conic.*

Let (x', y') be the given point, and let a line be drawn through it making an angle θ with the axis of x. The point which is at a distance r along the line from (x', y') is $(x' + r\cos\theta,\ y' + r\sin\theta)$, the axes being supposed to be rectangular; and, if this point be on the conic given by the general equation, we have

$$a(x' + r\cos\theta)^2 + 2h(x' + r\cos\theta)(y' + r\sin\theta) + b(y' + r\sin\theta)^2$$
$$+ 2g(x' + r\cos\theta) + 2f(y' + r\sin\theta) + c = 0,$$

or $$r^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)$$
$$+ 2r\cos\theta(ax' + hy' + g) + 2r\sin\theta(hx' + by' + f) + \phi(x', y') = 0.$$

The roots of this quadratic equation are the two values of r required.

185. If the point (x', y') be the middle point of the chord intercepted by the conic on the line, the two values of r, given by the quadratic equation in the preceding Article, will be equal in magnitude and opposite in sign; hence the coefficient of r must vanish; thus

$$(ax' + hy' + g)\cos\theta + (hx' + by' + f)\sin\theta = 0.$$

If the chords are always drawn in a fixed direction, so that θ is constant, the above equation gives us the relation satisfied by the co-ordinates x', y' of the middle point of any chord.

The locus of the middle points of chords of the conic which make an angle θ with the axis of x is therefore a straight line. [See Art. 182.]

186. The rectangle of the segments of the chord which passes through the point (x', y') and makes an angle θ with the axis of x, is the product of the two values of r given by the quadratic equation in Art. 184; and is equal to

$$\frac{\phi(x', y')}{a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta}.$$

Cor. 1. If through the same point (x', y') another chord be drawn making an angle θ' with the axis of x, the rectangle of the segments of this chord will be

$$\frac{\phi(x', y')}{a\cos^2\theta' + 2h\sin\theta'\cos\theta' + b\sin^2\theta'}.$$

Hence we see that the ratio of the rectangles of the segments of two chords of a conic drawn in given directions through the same point is constant for all points, including the centre of the conic, so that the ratio is equal to the ratio of the squares of the parallel diameters of the conic.

Cor. 2. The ratio of the two tangents drawn to a conic from any point is equal to the ratio of the parallel diameters of the conic.

Cor. 3. If through the point (x'', y'') a chord be drawn also making an angle θ with the axis of x, the rectangle of the segments of this chord will be

$$\frac{\phi(x'', y'')}{a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta}.$$

Hence the ratio of the rectangles of the segments of any two parallel chords drawn through two fixed points (x', y') and (x'', y'') is constant and equal to the ratio of $\phi(x', y')$ to $\phi(x'', y'')$.

Cor. 4. If a circle cut a conic in four points P, Q, R, S, the line PQ joining any two of the points and the line RS joining the other two make equal angles with an axis of the conic.

For, if PQ and RS meet in T, the rectangles $TP \cdot TQ$ and $TR \cdot TS$ are equal since the four points are on a circle. Therefore by Cor. 1, the parallel diameters of the conic are equal; and hence they must be equally inclined to an axis of the conic.

Ex. 1. *If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the four points of intersection of a circle and an ellipse, then will $\alpha+\beta+\gamma+\delta=2n\pi$.*

The equations of the lines joining α, β and γ, δ are

$$\frac{x}{a}\cos\tfrac{1}{2}(\alpha+\beta)+\frac{y}{b}\sin\tfrac{1}{2}(\alpha+\beta)=\cos\tfrac{1}{2}(\alpha-\beta),$$

and $$\frac{x}{a}\cos\tfrac{1}{2}(\gamma+\delta)+\frac{y}{b}\sin\tfrac{1}{2}(\gamma+\delta)=\cos\tfrac{1}{2}(\gamma-\delta).$$

These two chords are equally inclined to the axis by Cor. 4: therefore $\tan\frac{1}{2}(\alpha+\beta)=-\tan\frac{1}{2}(\gamma+\delta)$, or $\frac{1}{2}(\alpha+\beta)=n\pi-\frac{1}{2}(\gamma+\delta)$; therefore
$$\alpha+\beta+\gamma+\delta=2n\pi.$$

Ex. 2. *A focal chord of a conic varies as the square of the parallel diameter.* [See Art. 161.]

Ex. 3. *If a triangle circumscribe a conic the three lines from the angular points of the triangle to the points of contact of the opposite sides will meet in a point.*

Let the angular points be A, B, C and the points of contact of the opposite sides of the triangle be A', B', C'; also let r_1, r_2, r_3 be the semi-diameters of the conic parallel to the sides of the triangle. Then

$BA' : BC' = r_1 : r_3$; $CB' : CA' = r_2 : r_1$; and $AC' : AB' = r_3 : r_2$.

Hence $$BA' . CB' . AC' = BC' . AB' . CA',$$
which shews that the three lines meet in a point.

Ex. 4. *If a conic cut the three sides of a triangle ABC in the points A' and A'', B' and B'', C' and C'' respectively, then will*
$$BA' . BA'' . CB' . CB'' . AC' . AC'' = BC' . BC'' . CA' . CA'' . AB' . AB''.$$
(*Carnot's Theorem.*)

[$BA' . BA'' : BC' . BC'' = r_1^2 : r_2^2$, and so for the others; r_1, r_2, r_3 being the semi-diameters of the conic parallel to the sides of the triangle.]

Ex. 5. *If a conic touch all the sides of a polygon $ABCD$...... the points of contact of the sides AB, BC...... being P, Q, R, S......; then will $AP . BQ . CR . DS$...... be equal to $PB . QC . RD$......*

187. If S be written for shortness instead of the left-hand side of the equation
$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$
and S' be written instead of the left-hand side of the equation
$$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0,$$
then $S-\lambda S'=0$ is the equation of a conic which passes through the points common to the two conics $S=0$, $S'=0$.

For, the equation $S-\lambda S'=0$ is of the second degree, and therefore represents some conic. Also if any point be on both the given conics, its co-ordinates will satisfy both the equations $S=0$ and $S'=0$, and therefore also the equation $S-\lambda S'=0$.

By giving a suitable value to λ, the conic $S-\lambda S'=0$ can be made to satisfy any one other condition.

If the conic $S'=0$ really be two straight lines whose equations are $lx+my+n=0$ and $l'x+m'y+n'=0$, which for shortness we will call $u=0$, and $v=0$, then $S-\lambda uv=0$ will, for all values of λ, be the equation of a conic passing through the points where $S=0$ is cut by the lines $u=0$ and $v=0$.

If now the line $v=0$ be supposed to move up to and ultimately coincide with the line $u=0$, the equation $S-\lambda u^2=0$ will, for all values of λ, represent a conic which cuts the conic $S=0$ in two pairs of coincident points, where $S=0$ is met by the line $u=0$. That is to say $S-\lambda u^2=0$ is a conic *touching* $S=0$ at the *two* points where $S=0$ is cut by $u=0$. (ω).

Ex. 1. *All conics through the points of intersection of two rectangular hyperbolas are rectangular hyperbolas.*

If $S=0$, $S'=0$ be the equations of two rectangular hyperbolas, all conics through their points of intersection are included in the equation $S-\lambda S'=0$. Now the sum of the coefficients of x^2 and y^2 in $S-\lambda S'=0$ will be zero, since that sum is zero in S and also in S', the axes being at right angles. This proves the proposition. [Art. 175.]

The following are particular cases of the above.

(i) If two rectangular hyperbolas intersect in four points, the line joining any two of the points is perpendicular to the line joining the other two. (For the pair of lines is a conic through the points of intersection.) (ii) If a rectangular hyperbola pass through the angular points of a triangle it will also pass through the orthocentre. (For, if A, B, C be the angular points, and the perpendicular from A on BC cut the conic in D; then the pair of lines AD, BC is a rectangular hyperbola, since these lines are at right angles; therefore the pair BD, AC is also a rectangular hyperbola, that is to say the lines are at right angles.)

Ex. 2. *If two conics have their axes parallel a circle will pass through their points of intersection.*

Take axes parallel to the axes of the conics, their equations will then be $ax^2+by^2+2gx+2fy+c=0$,
and $a'x^2+b'y^2+2g'x+2f'y+c'=0$.
The conic $ax^2+by^2+2gx+2fy+c+\lambda(a'x^2+b'y^2+2g'x+2f'y+c')=0$ will go through their intersections. But this will be a circle, if we choose λ so that $a+\lambda a'=b+\lambda b'$, and this is clearly always possible.

Ex. 3. *If TP, TQ and $T'P'$, $T'Q'$ be tangents to an ellipse, a conic will pass through the six points T, P, Q, T', P', Q'.*

Let the conic be $ax^2+by^2=1$, and let T be (x', y') and T' be (x'', y''). The equations of PQ and $P'Q'$ will be $axx'+byy'-1=0$ and $axx''+byy''-1=0$. The conic

$$\lambda(ax^2+by^2-1)-(axx'+byy'-1)(axx''+byy''-1)=0$$

will always pass through the four points P, Q, P', Q'. It will also pass through T if λ be such that

$$\lambda(ax'^2+by'^2-1)-(ax'^2+by'^2-1)(ax'x''+by'y''-1)=0,$$

or if $$\lambda=ax'x''+by'y''-1.$$

The symmetry of this result shews that the conic will likewise pass through T'.

Ex. 4. *If two chords of a conic be drawn through two points on a diameter equidistant from the centre, any conic through the extremities of those chords will be cut by that diameter in points equidistant from the centre.*

Take the diameter and its conjugate for axes, then the equation of the conic will be $ax^2+by^2=1$. Let the equations of the chords be $y-m(x-c)=0$ and $y-m'(x+c)=0$. Then the equation of any conic through their extremities is given by

$$ax^2+by^2-1-\lambda\{y-m(x-c)\}\{y-m'(x+c)\}=0.$$

The axis of x cuts this in points given by $ax^2-1-\lambda mm'(x^2-c^2)=0$, and these two values of x are clearly equal and opposite whatever λ, m and m' may be.

As a particular case, if PSQ and $P'S'Q'$ be two focal chords of a conic, the lines PP' and QQ' cut the axis in points equidistant from the centre.

188. *To find the equation of the pair of tangents drawn from any point to a conic.*

Let the equation of the conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0\text{......(i).}$$

If (x', y') be the point from which the tangents are drawn, the equation of the chord of contact will be

$$axx' + h(xy' + yx') + byy' + g(x + x') + f(y + y') + c = 0.$$

The equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$
$$= \lambda \{axx' + h(xy' + yx') + byy' + g(x + x') + f(y + y') + c\}^2 \quad \ldots\ldots\text{(ii)}$$

represents a conic touching the original conic at the two points where it is met by the chord of contact. The two tangents are a conic which touches at these two points and which also passes through the point (x', y') itself. The equation (ii) will therefore be the equation required if λ be so chosen that (x', y') is on (ii); that is, if

$$ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c$$
$$= \lambda \{ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c\}^2.$$

Therefore

$$1 = \lambda \{ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c\} = \lambda\phi(x', y').$$

Substituting this value of λ in (ii) we have

$$(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)\,\phi(x', y')$$
$$= \{axx' + h(xy' + yx') + byy' + g(x + x') + f(y + y') + c\}^2,$$

which is the required equation. (ω).

The above equation may be found in the following manner.

Let TQ, TQ' be the two tangents from (x', y'), let $P(x, y)$ be any point on TQ, and let TN, PM be the perpendiculars from T and P on the chord of contact QQ'.

Then $$\frac{PQ^2}{TQ^2} = \frac{PM^2}{TN^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

But [Art. 186, Cor. 3] $$\frac{PQ^2}{TQ^2} = \frac{\phi(x, y)}{\phi(x', y')},$$

and [Art. 31]

$$\frac{PM^2}{TN^2} = \frac{\{axx' + h(xy' + yx') + byy' + g(x + x') + f(y + y') + c\}^2}{\{ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c\}^2};$$

therefore from (i) we have

$$\phi(x, y)\,\phi(x', y') = \{axx' + h(xy' + yx') + byy' + g(x + x') + f(y + y') + c\}^2.$$

189. *To find the equation of the director-circle of a conic.*

The equation of the tangents drawn from (x', y') to the conic given by the general equation is

$$(ax^2+2hxy+by^2+2gx+2fy+c)\,\phi(x',\,y')$$
$$=\{axx'+h(xy'+yx')+byy'+g(x+x')+f(y+y')+c\}^2.$$

The two tangents will be at right angles to one another if the sum of the coefficients of x^2 and y^2 in the above equation is zero. This requires that

$$(a+b)(ax'^2+2hx'y'+by'^2+2gx'+2fy'+c)$$
$$-(ax'+hy'+g)^2-(hx'+by'+f)^2=0.$$

The point $(x',\,y')$ is therefore on the circle whose equation is

$$(ab-h^2)(x^2+y^2)+2x(gb-fh)+2y(fa-hg)+c(a+b)$$
$$-f^2-g^2=0,$$

or $$Cx^2+Cy^2-2Gx-2Fy+A+B=0\ \ldots\ldots\text{(i)},$$

where A, B, C, F, G, H mean the same as in Art. 178.

If $h^2-ab=0$, the equation reduces to

$$2x(bg-fh)+2y(fa-hg)+c(a+b)-f^2-g^2=0,$$

or $$2Gx+2Fy-A-B=0\ldots\ldots\ldots\ldots\text{(ii)}.$$

The conic in this case is a parabola, and (ii) is the equation of its *directrix*.

Ex. 1. Trace the curve $11x^2+24xy+4y^2-2x+16y+11=0$, and shew that the equation of the director-circle is $x^2+y^2+2x-2y=1$.

Ex. 2. Shew that the equation of the directrix of the parabola
$$x^2+2xy+y^2-4x+8y-6=0 \text{ is } 3x-3y+8=0.$$

190. *To shew that a central conic has four and only four foci, two of which are real and two imaginary.*

Let the equation of the conic be

$$ax^2+by^2-1=0\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Let $(x',\,y')$ be a focus, and let $x\cos\alpha+y\sin\alpha-p=0$ be the equation of the corresponding directrix; then if e be the eccentricity of the conic, the equation will be

$$(x-x')^2+(y-y')^2-e^2(x\cos\alpha+y\sin\alpha-p)^2=0\ldots\text{(ii)}.$$

Since (i) and (ii) represent the same curve, and the coefficient of xy is zero in (i), the coefficient of xy must be zero in (ii); hence α is 0 or $\frac{\pi}{2}$.

Hence a directrix is parallel to one or other of the axes.

Let $\alpha=0$, then since the coefficients of x and y are zero in (i), we have $y'=0$ and $x'=e^2p$.

Also, by comparing the other coefficients in (i) and (ii), we have

$$\frac{a}{1-e^2}=\frac{b}{1}=\frac{-1}{x'^2-e^2p^2};$$

$$\therefore\ e=\sqrt{\left(1-\frac{a}{b}\right)}\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)},$$

$$apx'=1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)},$$

and

$$x'^2=\frac{1}{a}-\frac{1}{b}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}.$$

From (v) we see that there are two foci on the axis of x whose distances from the centre are $\pm\sqrt{\left(\frac{1}{a}-\frac{1}{b}\right)}$. From (iv) we see that a directrix is the polar of the corresponding focus.

If $\alpha=\frac{\pi}{2}$, we can shew in a similar manner that there are two foci on the axis of y whose distances from the centre are $\pm\sqrt{\left(\frac{1}{b}-\frac{1}{a}\right)}$. Of the two pairs of foci one is clearly real and the other imaginary, whatever the values of a and b (supposed real) may be.

The eccentricity of a conic referred to a focus on the axis of x is from (iii) equal to $\sqrt{\left(1-\frac{a}{b}\right)}$; the eccentricity referred to a focus on the axis of y will similarly be $\sqrt{\left(1-\frac{b}{a}\right)}$. If the curve be an ellipse a and b have the same sign, and one of these eccentricities is real and the other imaginary. If however the curve be an hyperbola, a and b have different signs and *both* eccentricities are real.

In any conic, if e_1 and e_2 be the two eccentricities, we have

$$\frac{1}{e_1^2}+\frac{1}{e_2^2}=\frac{a}{a-b}+\frac{b}{b-a}=1.$$

191. *To find the eccentricity of a conic given by the general equation of the second degree.*

By changing the axes we can reduce the conic to the form

$$\alpha x^2+\beta y^2+\gamma=0 \text{..................(i)}.$$

If e be one of the eccentricities of the conic,

$$\alpha=\beta(1-e^2) \text{..................(ii)}.$$

But [Art. 52], we know that

$$\alpha+\beta=a+b \text{..................(iii)},$$

and

$$\alpha\beta=ab-h^2 \text{..................(iv)}.$$

Eliminating α and β from the equations (ii), (iii) and (iv), we have

$$\frac{(2-e^2)^2}{1-e^2}=\frac{(a+b)^2}{ab-h^2},$$

or

$$e^4+\frac{(a-b)^2+4h^2}{ab-h^2}(e^2-1)=0 \text{.............(v)}.$$

If the curve is an ellipse, $ab-h^2$ is positive, and one value of e^2 is positive and the other negative. The real value of e is the eccentricity of the ellipse with reference to one of the real foci, and the imaginary value is the eccentricity with reference to one of the imaginary foci.

If the curve is an hyperbola both values of e^2 are positive, and therefore both eccentricities are real, as we found in Art. 190; we must therefore distinguish between the two eccentricities.

The signs of α and β in (i) are different when the curve is an hyperbola; and, if the sign of α be different from that of γ, the real foci will lie on the axis of x. Hence to find the eccentricity with reference to a real focus; obtain the values of α and β from (iii) and (iv), then (ii) will give the eccentricity required, if we take for α that value whose sign is different from the sign of γ.

Ex. Find the eccentricity of the conic whose equation is

$$x^2 - 4xy - 2y^2 + 10x + 4y = 0.$$

The equation referred to the centre is $x^2 - 4xy - 2y^2 - 1 = 0$. This will become $\alpha x^2 + \beta y^2 - 1 = 0$, where $\alpha + \beta = -1$ and $\alpha\beta = -6$. Hence $\alpha = 2$, $\beta = -3$. The eccentricity with reference to a real focus is given by $2 = -3(1 - e^2)$; therefore $e = \sqrt{\frac{5}{3}}$.

192. *To find the foci of a conic.*

Let (x', y') be one of the foci of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{........(i)}.$$

The corresponding directrix of the conic is the polar of (x', y'); therefore its equation is

$$x(ax' + hy' + g) + y(hx' + by' + f) + gx' + fy' + c = 0.$$

The equation of the conic may therefore be written in the form

$$(x - x')^2 + (y - y')^2 - \lambda\{x(ax' + hy' + g) + y(hx' + by' + f) + gx' + fy' + c\}^2 = 0 \quad \text{......(ii)}.$$

Since (i) and (ii) represent the same curve, the coefficients in (ii) must be equal to the corresponding coefficients in (i) multiplied by some constant. We have therefore

$$\begin{aligned}
1 - \lambda(ax' + hy' + g)^2 &= ka,\\
-\lambda(ax' + hy' + g)(hx' + by' + f) &= kh,\\
1 - \lambda(hx' + by' + f)^2 &= kb,\\
-x' - \lambda(ax' + hy' + g)(gx' + fy' + c) &= kg,\\
-y' - \lambda(hx' + by' + f)(gx' + fy' + c) &= kf,
\end{aligned}$$

and $$x'^2 + y'^2 - \lambda(gx' + fy' + c)^2 = kc.$$

From the first three of the above equations we have

$$\frac{(ax' + hy' + g)^2 - (hx' + by' + f^2)}{a - b},$$

$$= \frac{(ax' + hy' + g)(hx' + by' + f)}{h} \quad \text{...(iii)}.$$

Multiply the fourth and fifth equations by x', y' respectively and add them to the sixth; then, comparing with the second, after rejecting the factor $gx'+fy'+c$, we get

$$x'(ax'+hy'+g)+y'(hx'+by'+f)+gx'+fy'+c$$
$$=\frac{(ax'+hy'+g)(hx'+by'+f)}{h},$$

or $$\phi(x', y')=\frac{(ax'+hy'+g)(hx'+by'+f)}{h}\ldots\text{(iv)}.$$

The four foci are therefore from (iii) and (iv) the four points of intersection of the two conics

$$\frac{(ax+hy+g)^2-(hx+by+f)^2}{a-b}=\frac{(ax+hy+g)(hx+by+f)}{h}$$
$$=\phi(x, y).$$

193. The equation of a conic referred to a focus as origin is $x^2+y^2=e^2(x\cos\alpha+y\sin\alpha-p)^2$.

Either of the lines $x\pm\sqrt{-1}\,y=0$ meets the conic in *coincident points.*

Hence the tangents from the focus to the conic are the imaginary lines $x\pm y\sqrt{-1}=0$, or as one equation

$$x^2+y^2=0.$$

Since the equation of the tangents from a focus is independent of the position of the directrix, it follows that if conics have one focus common they have two imaginary tangents common, and that confocal conics have four common tangents.

Now if the origin and axes of co-ordinates be changed in any manner, the equation of the tangents from a focus will be changed from

$$x^2+y^2=0 \text{ to } x^2+y^2+2gx+2fy+c=0.$$

Hence the equation of the tangents to a conic from a focus satisfies the conditions for a circle.

We may therefore find the foci of a conic in the following manner.

The equation of the tangents from (x', y') to the conic $\phi(x, y) = 0$ is

$$(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)\,\phi(x', y')$$
$$= \{ax'x + h(x'y + y'x) + by'y + g(x + x') + f(y + y') + c\}^2.$$

If (x', y') be a focus of the conic, this equation satisfies the conditions for a circle, viz. that the coefficients of x^2 and y^2 are equal, and that the coefficient of xy is zero.

Hence we have

$$a\phi(x', y') - (ax' + hy' + g)^2 = b\phi(x', y') - (hx' + by' + f)^2,$$

and $$h\phi(x', y') = (ax' + hy' + g)(hx' + by' + f).$$

The foci are therefore the points given by

$$\frac{(ax + hy + g)^2 - (hx + by + f)^2}{a - b}$$
$$= \frac{(ax + hy + g)(hx + by + f)}{h} = \phi(x, y).$$

The equations giving the foci may be written

$$\frac{\left(\frac{d\phi}{dx}\right)^2 - \left(\frac{d\phi}{dy}\right)^2}{a - b} = \frac{\frac{d\phi}{dx}\,\frac{d\phi}{dy}}{h} = 4\phi.$$

194. *To find the equation of the axes of a conic.*

The axes of a conic bisect the angles between the asymptotes, and the asymptotes are parallel to the lines given by the equation $ax^2 + 2hyx + by^2 = 0$ [Art. 174]. Hence [Art. 39] the axes are the straight lines through the centre of the conic parallel to the lines given by the equation

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}.$$

We may also find the equation of the axes as follows.

If a point P be on an axis of the conic, the line joining P to the centre of the conic is perpendicular to the polar of P.

Let x', y' be the co-ordinates of P, then the equation of the polar of P is

$$x(ax' + hy' + g) + y(hx' + by' + f) + gx' + fy' + c = 0 \ldots \text{(i)}.$$

The equation of any line through the centre of the conic is

$$ax+hy+g+\lambda(hx+by+f)=0 \quad \text{.........(ii)}.$$

Since (ii) is perpendicular to (i), we have

$$(a+\lambda h)(ax'+hy'+g)+(h+\lambda b)(hx'+by'+f)=0 \quad \text{...(iii)}.$$

Since (ii) passes through (x', y'), we have

$$ax'+hy'+g+\lambda(hx'+by'+f)=0 \quad \text{.......(iv)}.$$

Eliminate λ from (iii) and (iv), and we see that (x', y') must be on the conic

$$\frac{(ax+hy+g)^2-(hx+by+f)^2}{a-b}=\frac{(ax+hy+g)(hx+by+f)}{h},$$

which is the equation required.

The equation of the axes may also be deduced from Article 192 or 193; for one of the conics on which we have found that the foci lie passes through the centre, and therefore must be the axes.

Ex. 1. Shew that all conics through the four foci of a conic are rectangular hyperbolas.

Ex. 2. Prove that the foci of the conic whose equation is

$$ax^2+2hxy+by^2=1,$$

lie on the curves

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}=\frac{1}{h^2-ab}.$$

Ex. 3. Shew that the real foci of the conic

$x^2-6xy+y^2-2x-2y+5=0$ are $(1, 1)$ and $(-2, -2)$.

Ex. 4. The co-ordinates of the real foci of $2x^2-8xy-4y^2-4y+1=0$ are

$$\left(0, \frac{1}{2}\right) \text{ and } \left(-\frac{2}{3}, -\frac{5}{6}\right).$$

Ex. 5. The focus of the parabola $x^2+2xy+y^2-4x+8y-6=0$ is the point $(-\frac{1}{3}, -\frac{2}{3})$.

Ex. 6. Shew that the product of the perpendiculars from the two imaginary foci of an ellipse on any tangent to the curve is equal to the square of the semi-major axis.

Ex. 7. Shew that the foot of the perpendicular from an imaginary focus of an ellipse on the tangent at any point lies on the circle described on the minor axis as diameter.

Ex. 8. If a circle have double contact with an ellipse, shew that the tangent to the circle from any point on the ellipse varies as the distance of that point from the chord of contact.

195. *To find the equation of a conic when the axes of co-ordinates are the tangent and normal at any point.*

The most general form of the equation of a conic is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Since the origin is on the curve, the co-ordinates (0, 0) will satisfy the equation, and therefore $c = 0$.

The line $y = 0$ meets the curve where $ax^2 + 2gx = 0$. If $y = 0$ is the tangent at the origin, *both* the values of x given by the equation $ax^2 + 2gx = 0$ must be zero; therefore $g = 0$.

Hence the most general form of the equation of a conic, when referred to a tangent and the corresponding normal as axes of x and y respectively, is

$$ax^2 + 2hxy + by^2 + 2fy = 0.$$

Ex. 1. *All chords of a conic which subtend a right angle at a fixed point O on the conic, cut the normal at O in a fixed point.*

Take the tangent and normal at O for axes; then the equation of the conic will be

$$ax^2 + 2hxy + by^2 + 2fy = 0.$$

Let the equation of PQ, one of the chords, be $lx + my - 1 = 0$. The equation of the lines OP, OQ will be [Art. 38]

$$ax^2 + 2hxy + by^2 + 2fy\,(lx + my) = 0 \text{ ....(i).}$$

But OP, OQ are at right angles to one another, therefore the sum of the coefficients of x^2 and y^2 in (i) is zero. Hence we have $a + b + 2fm = 0$; which shews that m is constant, and m is the reciprocal of the intercept on the normal.

Ex. 2. *If any two chords OP, OQ of a conic make equal angles with the tangent at O, the line PQ will cut that tangent in a fixed point.*

196. The equation of the normal at any point (x', y') of the conic whose equation is $ax^2 + by^2 = 1$ is

$$\frac{x - x'}{ax'} = \frac{y - y'}{by'}.$$

This will pass through the point (h, k) if

$$\frac{h-x'}{ax'}=\frac{k-y'}{by'},$$

i.e. if $$x'y'(a-b)+bhy'-akx'=0.$$

Therefore the feet of the normals which pass through a particular point (h, k) are on the conic

$$xy(a-b)+bhy-akx=0 \text{............}(i).$$

The four real or imaginary points of intersection of the conic (i) and the original conic are the points the normals at which pass through the point (h, k).

The conic (i) is clearly a rectangular hyperbola whose asymptotes are parallel to the axes of co-ordinates, that is to the axes of the original conic. It also passes through the centre of that conic, and through the point (h, k) itself.

197. If the normals at the extremities of the two chords $lx+my-1=0$ and $l'x+m'y-1=0$ meet in the point (h, k), then, for some value of λ, the conic

$$ax^2+by^2-1-\lambda(lx+my-1)(l'x+m'y-1)=0\text{...}(i),$$

which, for all values of λ, passes through the four extremities of the two chords, will [Art. 196] be the same as

$$xy(a-b)+bhy-akx=0\text{............}(ii).$$

The coefficients of x^2 and y^2, and the constant term are all zero in this last equation, and therefore they must be zero in the preceding.

We have therefore

$$a-\lambda ll'=0,\ b-\lambda mm'=0, \text{ and } 1+\lambda=0.$$

Hence, if the normals at the ends of the chords $lx+my-1=0$ and $l'x+m'y-1=0$ meet in a point, we have

$$\frac{ll'}{a}=\frac{mm'}{b}=-1\text{..............}(iii).$$

198. By the preceding Article we see that normals to the ellipse whose axes are $2a$, $2b$ at the extremities of the

chords whose equations are

$$lx + my - 1 = 0, \text{ and } l'x + m'y - 1 = 0,$$

will meet in a point, if

$$a^2 ll' = b^2 mm' = -1 \ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

If the eccentric angles of these four points be α, β and γ, δ, the equations of the chords will be

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$$

and $$\frac{x}{a}\cos\frac{\gamma+\delta}{2} + \frac{y}{b}\sin\frac{\gamma+\delta}{2} = \cos\frac{\gamma-\delta}{2}.$$

We have therefore, by comparing with (i),

$$\cos\frac{\alpha+\beta}{2}\cos\frac{\gamma+\delta}{2} + \cos\frac{\alpha-\beta}{2}\cos\frac{\gamma-\delta}{2} = 0,$$

and $$\sin\frac{\alpha+\beta}{2}\sin\frac{\gamma+\delta}{2} + \cos\frac{\alpha-\beta}{2}\cos\frac{\gamma-\delta}{2} = 0.$$

By subtraction, we have

$$\cos\frac{\alpha+\beta+\gamma+\delta}{2} = 0,$$

whence $$\alpha+\beta+\gamma+\delta = (2n+1)\,\pi \ldots\ldots\ldots\text{(ii)}.$$

Also the first equation gives

$$\cos\frac{\alpha+\beta+\gamma+\delta}{2} + \cos\frac{\alpha+\beta-\gamma-\delta}{2} + \cos\frac{\alpha+\gamma-\beta-\delta}{2} + \cos\frac{\alpha+\delta-\beta-\gamma}{2} = 0,$$

and, using the condition (ii), this becomes

$$\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) = 0 \ldots\text{(iii)}.$$

Ex. 1. *If ABC be a maximum triangle inscribed in an ellipse, the normals at A, B, C will meet in a point.*

The eccentric angles will be α, $\alpha+\frac{2\pi}{3}$, and $\alpha+\frac{4\pi}{3}$ [Art. 138]. The condition that the normals meet in a point is [Art. 198 (iii)]

$$\sin 2\alpha + \sin\left(2\alpha + \frac{2\pi}{3}\right) + \sin\left(2\alpha + \frac{4\pi}{3}\right) = 0,$$

which is clearly true.

Ex. 2. *The normals to a central conic at the four points P, Q, R, S meet in a point, and the circle through P, Q, R cuts the conic again in S'; shew that SS' is a diameter of the conic.*

SS' will be a diameter of the conic if RS and RS' are parallel to conjugate diameters [Art. 134].

Now if PQ be $lx + my - 1 = 0$, RS will be $\frac{a}{l}x + \frac{b}{m}y + 1 = 0$ [Art. 197]; also RS' will be parallel to $lx - my = 0$, since P, Q, R, S' are on a circle; hence SS' is a diameter, for [Art. 182] $lx - my = 0$, and $\frac{a}{l}x + \frac{b}{m}y = 0$ are conjugate diameters of $ax^2 + by^2 = 1$.

[The proposition may also be obtained from Art. 198 (ii), and Art. 186, Ex. (1).]

Ex. 3. *If the normals to an ellipse at A, B, C, D meet in a point, the axis of a parabola through A, B, C, D is parallel to one or other of the equi-conjugates.*

If h, k be the point where the normals meet, A, B, C, D are the four points of intersection of the conics

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ and } xy\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \frac{hy}{b^2} - \frac{kx}{a^2} = 0.$$

All conics through the intersections are included in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 - \lambda\left\{xy\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \frac{hy}{b^2} - \frac{kx}{a^2}\right\} = 0.$$

If this be a parabola the terms of the second degree must be a perfect square, and therefore must be the square of $\frac{x}{a} \pm \frac{y}{b}$. The equation of every such parabola is therefore of the form $\left(\frac{x}{a} \pm \frac{y}{b}\right)^2 + Ax + By + C = 0$. Their axes are therefore [Art. 172] parallel to one or other of the lines $\frac{x}{a} \pm \frac{y}{b} = 0$.

Ex. 4. *The perpendicular from any point P on its polar with respect to a conic passes through a fixed point O; prove (α) that the locus of P is a rectangular hyperbola, (β) that the circle circumscribing the triangle which the polar of P cuts off from the axes always passes through a fixed point O', (γ) that a parabola whose focus is O' will touch the axes and all such polars, (δ) that the directrix of this parabola is CO, where C is the centre of the conic, and (ϵ) that O and O' are interchangeable.*

Let the equation of the conic be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and let (h, k) be the co-ordinates of the fixed point O.

If the co-ordinates of any point P be (x', y'), the equation of the line through P perpendicular to its polar with respect to the conic will be

$$\frac{x-x'}{\frac{x'}{a^2}}=\frac{y-y'}{\frac{y'}{b^2}},$$

or
$$\frac{a^2x}{x'}-\frac{b^2y}{y'}=a^2-b^2.$$

If this line pass through the point (h, k), we have

$$\frac{a^2h}{x'}-\frac{b^2k}{y'}=a^2-b^2 \text{......................} \text{(i)}.$$

From (i) we see that (x', y') is on a rectangular hyperbola ......(α).

The equation of the circle circumscribing the triangle cut off from the axes by the polar of (x', y') will be

$$x^2+y^2-\frac{a^2x}{x'}-\frac{b^2y}{y'}=0.$$

The circle will pass through the point $(\lambda h, -\lambda k)$ if

$$\lambda(h^2+k^2)=\frac{a^2h}{x'}-\frac{b^2k}{y'}.$$

Hence, if (x', y') satisfies the relation (i), we have

$$\lambda=\frac{a^2-b^2}{h^2+k^2}.$$

Hence the circles all pass through the point O' whose co-ordinates are

$$\frac{a^2-b^2}{h^2+k^2}h,\ \frac{b^2-a^2}{h^2+k^2}k \text{.........................} (\beta).$$

The point O' is on the circle circumscribing the triangle formed by the axes and any one of the polars; hence the parabola whose focus is O' and which touches the axes will touch every one of the polars............(γ).

The parabola touches the axes of the original conic, therefore the centre C is a point on the directrix of the parabola. Also the lines CO and CO' make equal angles with the axis of x, which is a tangent to the parabola; therefore O' being the focus, CO is the directrix..........(δ).

Since $CO' \,.\, CO=a^2-b^2$, and CO, CO' make equal angles with the axis of x, and are on the same side of the axis of y, the points O and O' are interchangeable ....(ϵ).

199. *Definition.* Two curves are said to be *similar* and *similarly situated* when radii vectores drawn to the first from a certain point O are in a constant ratio to

parallel radii vectores drawn to the second from another point O'.

Two curves are *similar* when radii drawn from two fixed points O and O' making a constant angle with one another are proportional.

The two fixed points O and O' may be called *centres of similarity.*

200. *If one pair of centres of similarity exist for two curves, then there will be an infinite number of such pairs.*

Let O, O' be the given centres of similarity, and let OP, $O'P'$ be any pair of parallel radii. Take C any point whatever, and draw $O'C'$ parallel to OC and in the ratio $O'P' : OP$. Then, from the similar triangles COP, and $C'O'P'$ we see that CP is parallel to $C'P'$ and in a constant ratio to it; which proves that C, C' are centres of similarity.

201. *If two central conics be similar the centres of the two curves will be centres of similarity.*

Let O and O' be two centres of similarity. Draw any chord POQ of the one, and the corresponding chord $P'O'Q'$ of the other. Then by supposition $PO \,.\, OQ : P'O' \,.\, O'Q'$ is constant for every pair of corresponding chords. But since O is a fixed point $PO \,.\, OQ$ is always in a constant ratio to the square of the diameter of the first conic which is parallel to it. The same applies to the other conic. Therefore corresponding diameters of the two conics are in a constant ratio to one another; this shews that the centres of the curves are centres of similarity.

202. *To find the conditions that two conics may be similar and similarly situated.*

By the preceding Article, their respective centres are centres of similarity.

Let the equations of the conics referred to those centres and parallel axes be

$$ax^2 + 2hxy + by^2 + c = 0,$$

and

$$a'x^2 + 2h'xy + b'y^2 + c' = 0;$$

or, in polar co-ordinates,

$$r^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta) + c = 0,$$

and $$r'^2\{a'\cos^2\theta + 2h'\sin\theta\cos\theta + b'\sin^2\theta\} + c' = 0.$$

If therefore $r^2 : r'^2$ be constant, we must have

$$\frac{a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta}{a'\cos^2\theta + 2h'\sin\theta\cos\theta + b'\sin^2\theta}$$

the same for all values of θ.

This requires that $\frac{a}{a'} = \frac{h}{h'} = \frac{b}{b'}$. Hence the asymptotes of the two conics are parallel. [This result may be obtained in the following manner: since $r : r'$ is constant, when one of the two becomes infinite, the other will also be infinite, which shews that the asymptotes are parallel.]

Conversely, if these conditions be satisfied, and if each fraction be equal to λ, then

$$\frac{r^2}{r'^2} = \frac{c}{\lambda c'};$$

therefore the ratio of corresponding radii is constant, and therefore the curves are similar.

If c and $\lambda c'$ have not the same sign the constant ratio is *imaginary*, and is zero or infinite if c or c' be zero.

The conditions of similarity are satisfied by the three curves whose equations are

$$xy = c,\ xy = 0, \text{ and } xy = -c.$$

Therefore an hyperbola, the conjugate hyperbola, and their asymptotes are three similar and similarly situated curves; the constant ratio being $\sqrt{-1}$ for the conjugate hyperbola, and zero for the straight lines.

These curves have not however the *same shape*. For similar curves to have the same shape the constant ratio must be *real* and *finite*.

203. *To find the condition that two conics may be similar although not similarly situated.*

We have seen that the centres of the two curves must be centres of similarity.

Let the equations of the curves referred to their respective centres be

$$ax^2 + 2hxy + by^2 + c = 0 \text{ ....(i)},$$

$$a'x^2 + 2h'xy + b'y^2 + c' = 0 \text{ ....(ii)},$$

and let the chord which makes an angle θ with the axis of x in the first be proportional, for all values of θ, to that which makes an angle $(\theta + \alpha)$ in the second. If the axes of the second conic be turned through the angle α, we shall then have radii of the two conics which make the same angle with the respective axes in a constant ratio.

Let the equation of the second conic become

$$A'x^2 + 2H'xy + B'y^2 + c' = 0.$$

Then, by the preceding Article, we must have

$$\frac{A'}{a} = \frac{H'}{h} = \frac{B'}{b};$$

therefore $$\frac{A' + B'}{a + b} = \frac{\sqrt{(A'B' - H'^2)}}{\sqrt{(ab - h^2)}}.$$

But [Art. 52] $A' + B' = a' + b'$, and $A'B' - H'^2 = a'b' - h'^2$; therefore the condition of similarity is

$$\frac{ab - h^2}{(a+b)^2} = \frac{a'b' - h'^2}{(a' + b')^2}.$$

The above shews that the angles between the asymptotes of similar conics are equal. [See Art. 174.]

This result may also be obtained in the following manner: since radii vectores of the two curves which are inclined to one another at a certain constant angle are in a constant ratio, it follows that the angle between the two directions which give infinite values for the one curve must be equal to the corresponding angle for the other, that is to say the angle between the asymptotes of the one conic is equal to the angle between the asymptotes of the other.

EXAMPLES ON CHAPTER X.

1. If Q and P be any two points, and C the centre of a conic; shew that the perpendiculars from Q and C on the polar of P with respect to the conic, are to one another in the same ratio as the perpendiculars from P and C on the polar of Q.

2. Two tangents drawn to a conic from any point are in the same ratio as the corresponding normals.

3. Find the loci of the fixed points of the examples in Article 195, for different positions of O on the conic.

4. POQ is one of a system of parallel chords of an ellipse, and O is the point on it such that $PO^2 + OQ^2$ is constant; shew that, for different positions of the chord, the locus of O is a concentric conic.

5. If O be any fixed point and OPP' any chord cutting a conic in P, P', and on this line a point D be taken such that $\frac{1}{OD^2} = \frac{1}{OP^2} + \frac{1}{OP'^2}$, the locus of D will be a conic whose centre is O.

6. If $OPP'QQ'$ is one of a system of parallel straight lines cutting one given conic in P, P' and another in Q, Q', and O is such that the ratio of the rectangles $OP \cdot OP'$ and $OQ \cdot OQ'$ is constant; shew that the locus of O is a conic through the intersections of the original conics.

7. POP', QOQ' are any two chords of a conic at right angles to one another through a fixed point O; shew that $\frac{1}{PO \cdot OP'} + \frac{1}{QO \cdot OQ'}$ is constant.

8. If a point be taken on the axis-major of an ellipse, whose abscissa is equal to $a\sqrt{\frac{a^2 - b^2}{a^2 + b^2}}$, prove that the sum of the squares of the reciprocals of the segments of any chord passing through that point is constant.

9. If PP' be any one of a system of parallel chords of a rectangular hyperbola, and A, A' be the extremities of the perpendicular diameter; PA and $P'A'$ will meet on a fixed circle. Shew also that the words rectangular hyperbola, and circle, can be interchanged.

10. If PSP' be any focal chord of a parabola and PM, $P'M'$ be perpendiculars on a fixed straight line, then will

$$\frac{PM}{PS} + \frac{P'M'}{P'S}$$

be constant.

11. Chords of a circle are drawn through a fixed point and circles are described on them as diameters; prove that the polar of the point with regard to any one of these circles touches a fixed parabola.

12. From a fixed point on a conic chords are drawn making equal intercepts, measured from the centre, on a fixed diameter; find the locus of the point of intersection of the tangents at their other extremities.

13. If (x', y') and (x'', y'') be the co-ordinates of the extremities of any focal chord of an ellipse, and $\bar{x}$, $\bar{y}$ be the co-ordinates of the middle point of the chord; shew that $y'\,y''$ will vary as $\bar{x}$. What does this become for a parabola?

14. S, H are two fixed points on the axis of an ellipse equidistant from the centre C; PSQ, PHQ' are chords through them, and the ordinate MQ is produced to R so that MR may be equal to the abscissa of Q'; shew that the locus of R is a rectangular hyperbola.

15. S, H are two fixed points on the axis of an ellipse equidistant from the centre, and PSQ, PHQ' are two chords of the ellipse; shew that the tangent at P and the line QQ' make angles with the axis whose tangents are in a constant ratio.

16. Two parallel chords of an ellipse, drawn through the foci, intersect the curve in points P, P' on the same side of the major axis, and the line through P, P' intersects the semi-axes CA, CB in U, V respectively: prove that $\frac{AC^4}{UC^2} + \frac{BC^4}{VC^2}$ is invariable.

17. From an external point two tangents are drawn to an ellipse; shew that if the four points where the tangents cut the axes lie on a circle, the point from which the tangents are drawn will lie on a fixed rectangular hyperbola.

18. Prove that the locus of the intersection of tangents to an ellipse which make equal angles with the major and minor axes respectively, but which are not at right angles, is a rectangular hyperbola whose vertices are the foci of the ellipse.

19. If a pair of tangents to a conic meet a fixed diameter in two points such that the *sum* of their distances from the centre is constant; shew that the locus of the point of intersection is a conic. Shew also that the locus of the point of intersection is a conic if the *product*, or if the *sum of the reciprocals* be constant.

20. Through O, the middle point of a chord AB of an ellipse, is drawn any chord POQ. The tangents at P and Q meet AB in S and T respectively. Prove that $AS = BT$.

21. Pairs of tangents are drawn to the conic $\alpha x^2 + \beta y^2 = 1$ so as to be always parallel to conjugate diameters of the conic $ax^2 + 2hxy + by^2 = 1$; shew that the locus of their intersection is

$$ax^2 + by^2 + 2hxy = \frac{a}{\alpha} + \frac{b}{\beta}.$$

22. PT, PT' are two tangents to an ellipse which meet the tangent at a fixed point Q in T, T'; find the locus of P (i) when the sum of the squares of QT and QT' is constant, and (ii) when the rectangle $QT \cdot QT'$ is constant.

23. O is a fixed point on the tangent at the vertex A of a conic, and P, P', are points on that tangent equally distant from O; shew that the locus of the point of intersection of the other tangents from P and P' is a straight line.

24. If from any point of the circle circumscribing a given square tangents be drawn to the circle inscribed in the same square, these tangents will meet the diagonals of the square in four points lying on a rectangular hyperbola.

25. Find the locus of the point of intersection of two tangents to a conic which intercept a constant length on a fixed straight line.

26. Two tangents to a conic meet a fixed straight line MN in P, Q: if P, Q be such that PQ subtends a right angle at a fixed point O, prove that the locus of the point of intersection of the tangents will be another conic.

27. The extremities of the diameter of a circle are joined to any point, and from that point two tangents are drawn to the circle; shew that the intercept on the perpendicular diameter between one line and one tangent is equal to that between the other line and the other tangent.

28. Triangles are described about an ellipse on a given base which touches the ellipse at P; if the base angles be equidistant from the centre, prove that the locus of their vertices is the normal at the other end of the diameter through P.

29. A parabola slides between rectangular axes; find the curve traced out by any point in its axis; and hence shew that the focus and vertex will describe curves of which the equations are

$$x^2y^2 = a^2(x^2 + y^2), \quad x^2y^2(x^2 + y^2 + 3a^2) = a^6,$$

$4a$ being the latus rectum of the parabola.

30. If the axes of co-ordinates be inclined to one another at an angle α, and an ellipse slide between them, shew that the equation of the locus of the centre is

$$\sin^2\alpha\,(x^2 + y^2 - p^2)^2 - 4\cos^2\alpha\,(x^2y^2\sin^2\alpha - q^4) = 0,$$

where p^2 and q^2 denote respectively the sum of the squares and the product of the semi-axes of the ellipse.

31. If OP, OQ are two tangents to an ellipse, and CP', CQ' the parallel semi-diameters, shew that

$$OP\,.\,OQ + CP'\,.\,CQ' = OS\,.\,OH,$$

S, H being the foci.

32. Through two fixed points P, Q straight lines APB, CQD are drawn at right angles to one another, to meet one given straight line in A, C and another given straight line perpen-

dicular to the former in B, D; find the locus of the point of intersection of AD, BC; and shew that, if the line joining P and Q subtend a right angle at the point of intersection of the given lines, the locus will be a rectangular hyperbola.

33. Prove that the locus of the foot of the perpendicular from a point on its polar with respect to an ellipse is a rectangular hyperbola, if the point lies on a fixed diameter of the ellipse.

34. The polars of a point P with respect to two concentric and co-axial conics intersect in a point Q; shew that if P moves on a fixed straight line, Q will describe a rectangular hyperbola.

35. Shew that if the polars of a point with respect to two given conics are either parallel or at right angles the locus of the point is a conic.

36. The line joining two points A and B meets the two lines OQ, OP in Q and P. A conic is described so that OP and OQ are the polars of A and B with respect to it. Shew that the locus of its centre is the line OR where R divides AB so that

$$AR : RB :: QR : RP.$$

37. Find the locus of the foci of all conics which have a common director-circle and one common point.

38. Shew that the locus of the foci of conics which have a given centre and touch two given straight lines is an hyperbola.

39. In the conic $ax^2 + 2hxy + by^2 = 2y$, the rectangle under the focal distances of the origin is

$$\frac{1}{ab - h^2}.$$

40. The focus of a conic is given, and the tangent at a given point; shew that the locus of the extremities of the conjugate diameter is a parabola of which the given focus is focus.

41. A series of conics have their foci on two adjacent sides of a given parallelogram and touch the other two sides; shew that their centres lie on a straight line.

42. If TP, TQ be tangents drawn from any point T to touch a conic in P and Q, and if S and H be the foci, then

$$\frac{ST^2}{SP \, . \, SQ} = \frac{HT^2}{HP \, . \, HQ}.$$

43. An ellipse is described concentric with and touching a given ellipse and passing through its foci; shew that the locus of the foci of the variable ellipse is a lemniscate.

44. Having given five points on a circle of radius a; shew that the centres of the five rectangular hyperbolas, each of which passes through four of the points, will all lie on a circle of radius $\frac{a}{2}$.

45. If a rectangular hyperbola have its asymptotes parallel to the axes of a conic, the centre of mean position of the four points of intersection is midway between the centres of the curves.

46. Three straight lines are drawn parallel respectively to the three sides of a triangle; shew that the six points in which they cut the sides lie on a conic.

47. If the normal at P to an ellipse meet the axes in the points G, G', and O be a point on it such that $\frac{2}{PO} = \frac{1}{PG} + \frac{1}{PG'}$; then will any chord through O subtend a right angle at P.

48. Through a fixed point O of an ellipse two chords OP, OP' are drawn; shew that, if the tangent at the other extremity O' of the diameter through O cut these lines produced in two points Q, Q' such that the rectangle $O'Q \, . \, O'Q'$ is constant, the line PP' will cut OO' in a fixed point.

49. A chord LM is drawn parallel to the tangent at any point P of a conic, and the line PR which bisects the angle LPM meets LM in R; prove that the locus of R is a hyperbola having its asymptotes parallel to the axes of the original conic.

50. A given centric conic is touched at the ends of a chord, drawn through a given point in its transverse axis, by another conic which passes through the centre of the former: prove that the locus of the centre of the latter conic is also a centric conic.

51. QQ' is a chord of an ellipse parallel to one of the equi-conjugate diameters, C being the centre of the ellipse; shew that the locus of the centre of the circle QCQ' for different positions of QQ' is an hyperbola.

52. A circle is drawn touching the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at any point and passing through the centre; shew that the locus of the foot of the perpendicular from the centre of the ellipse on the chord of intersection of the ellipse and circle is the ellipse $a^2x^2+b^2y^2=\dfrac{a^4b^4}{(a^2-b^2)^2}$.

53. Find the value of c in order that the hyperbola $2xy-c=0$ may touch the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$, and shew that the point of contact will be at an extremity of one of the equi-conjugate diameters of the ellipse.

Shew also that the polars of any point with respect to the two curves will meet on that diameter.

54. Shew that, if CD, EF be parallel chords of two circles which intersect in A and B, a conic section can be drawn through the six points A, B, C, D, E, F; and give a construction for the position of the major axis.

55. If the intersection P of the tangents to a conic at two of the points of its intersection with a circle lie on the circle, then the intersection P' of the tangents at the other two points will lie on the same circle. In this case find the relations connecting the positions of P and P' for a central conic, and deduce the relative positions of P and P' when the conic is a parabola.

56. If T, T' be any two points equidistant and on opposite sides of the directrix of a parabola, and TP, TQ be the tangents to the parabola from T, and $T'Q'$, $T'P'$ the tangents from T'; then will T, P, Q, T', P', Q' all lie on a rectangular hyperbola.

57. If a straight line cut two circles in A, A' and B, B' and if C, C' be the common points of the circles, and if O be any point; shew that the three circles OAA', OBB' and OCC' will have a common radical axis.

58. With a fixed point O for centre circles are described cutting a conic; shew that the locus of the middle points of the common chords of a circle and of the conic is a rectangular hyperbola.

59. With a fixed point O for centre any circle is described cutting a conic in four points real or imaginary; shew that the locus of the centres of all conics through these four points is a rectangular hyperbola, which is independent of the radius of the circle.

60. The normals at the ends of a focal chord of a conic intersect in O and the tangents in T; shew that TO produced will pass through the other focus.

61. If from any point four normals be drawn to an ellipse meeting an axis in G_1, G_2, G_3, G_4, then will

$$\frac{1}{CG_1}+\frac{1}{CG_2}+\frac{1}{CG_3}+\frac{1}{CG_4}=\frac{4}{CG_1+CG_2+CG_3+CG_4}.$$

62. If the normals to an ellipse at A, B, C, D meet in O, find the equation of the conic $ABCDO$, and shew that the locus of the centre of this conic for a fixed point O is a straight line if the ellipse be one of a set of co-axial ellipses.

63. The four normals to an ellipse at P, Q, R, S meet at O. Straight lines are drawn from P, Q, R, S such that they make the same angles with the axis of the ellipse as CP, CQ, CR, CS respectively: prove that these four lines meet in a point.

64. The normals at P, Q, R, S meet in a point O and lines are drawn through P, Q, R, S making with the axis of the ellipse the same angles as OP, OQ, OR, OS respectively: prove that these four lines meet in a point.

65. The normals at P, Q, R, S meet in a point; and P', Q', R', S' are the points of the auxiliary circle corresponding to P, Q, R, S respectively. If lines be drawn through P, Q, R, S parallel to $P'C$, $Q'C$, $R'C$ and $S'C$ respectively, shew that they will meet in a point.

66. If from a vertex of a conic perpendiculars be drawn to the four normals which meet in any point O, these lines will meet the conic again in four points on a circle.

67. Tangents are drawn from any point on the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=4$ to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; prove that the normals at the points of contact meet on the conic $a^2x^2+b^2y^2=\left(\frac{a^2-b^2}{2}\right)^2$.

68. If ABC be a triangle inscribed in an ellipse such that the tangents at the angular points are parallel to the opposite sides, shew that the normals at A, B, C will meet in some point O. Shew also that for different positions of the triangle the locus of O will be the ellipse $4a^2x^2+4b^2y^2=(a^2-b^2)^2$.

69. If the co-ordinates of the feet of the normals to $xy=a$ from the point (X, Y) be x_1, y_1; x_2, y_2; x_3, y_3; x_4, y_4; then $Y=y_1+y_2+y_3+y_4$, and $X=x_1+x_2+x_3+x_4$.

70. The locus of the point of intersection of the normals to a conic at the extremities of a chord which is parallel to a given straight line, is a conic.

71. Any tangent to the hyperbola $4xy=ab$ meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in points P, Q; shew that the normals to the ellipse at P and Q meet on a fixed diameter of the ellipse.

72. If four normals be drawn from the point O to the ellipse $b^2x^2+a^2y^2=a^2b^2$, and p_1, p_2, p_3, p_4 be the perpendiculars from the centre on the tangents to the ellipse drawn at the feet of these normals, then if

$$\frac{1}{p_1^2}+\frac{1}{p_2^2}+\frac{1}{p_3^2}+\frac{1}{p_4^2}=\frac{1}{c^2},$$

where c is a constant, the locus of O is a hyperbola.

73. Find the locus of a point when the sum of the squares of the four normals from it to an ellipse is constant.

74. The tangents to an ellipse at the feet of the normals which meet in (f, g) form a quadrilateral such that if (x', y'), (x'', y'') be any pair of opposite vertices $\frac{x'x''}{a^2}=\frac{y'y''}{b^2}=-1$, and that the equation of the line joining the middle points of the diagonals of the quadrilateral is $fx+gy=0$.

75. Tangents are drawn to an ellipse at four points which are such that the normals at those points co-intersect; and four rectangles are constructed each having two adjacent sides along

the axes of the ellipse, and one of those tangents for a diagonal. Prove that the distant extremities of the other diagonals lie in one straight line.

76. From a point P normals are drawn to an ellipse meeting it in A, B, C, D. If a conic can be described passing through A, B, C, D and a focus of the ellipse and touching the corresponding directrix, shew that P lies on one of two fixed straight lines.

77. If the normals at A, B, C, D meet in a point O, then will $SA \,.\, SB \,.\, SC \,.\, SD = k^2 \,.\, SO^2$, where S is a focus.

78. From any point four normals are drawn to a rectangular hyperbola; prove that the sum of the squares on these normals is equal to three times the square of the distance of the point from the centre of the hyperbola.

79. A chord is drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meeting the major axis in a point whose distance from the centre is $a\sqrt{\frac{a-b}{a+b}}$. At the extremities of this chord normals are drawn to the ellipse; prove that the locus of their point of intersection is a circle.

80. The product of the four normals drawn to a conic from any point is equal to the continued product of the two tangents drawn from that point and of the distances of the point from the asymptotes.

81. Find the equation of the conic to which the straight lines $(x + \lambda y)^2 - p^2 = 0$, and $(x + \mu y)^2 - q^2 = 0$ are tangents at the ends of conjugate diameters.

82. From any point T on the circle $x^2 + y^2 = c^2$, tangents TP, TQ are drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the circle TPQ cuts the ellipse again in $P'Q'$. Shew that the line $P'Q'$ always touches the ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{c^2}{(a^2 - b^2)^2}.$$

83. A focal chord of a conic cuts the tangents at the ends of the major axis in A, B; shew that the circle on AB as diameter has double contact with the conic.

84. $ABCD$ is any rectangle circumscribing an ellipse whose foci are S and H; shew that the circle ABS or ABH is equal to the auxiliary circle.

85. Any circle is described having its centre on the tangent at the vertex of a parabola, and the four common tangents of the circle and the parabola are drawn; shew that the sum of the tangents of the angles these lines make with the axis of the parabola is zero.

86. Tangents to an ellipse are drawn from any point on the auxiliary circle and intersect the directrix in four points: prove that two of these lie on a straight line passing through the centre, and find where the line through the other two points cuts the major axis.

87. If $u=0$, $v=0$ be the equation of two central conics, and u_0, v_0 the values of u, v at the centres C, C' of these conics respectively, shew that $u_0 v = v_0 u$ is the equation of the locus of the intersection of the lines CP, $C'P'$, where P, P' are two points, one on each curve, such that PP' is parallel to CC'. Examine the case where the conics are similar and similarly situated.

88. Two circles have double internal contact with an ellipse and a third circle passes through the four points of contact. If t, t', T be the tangents drawn from any point on the ellipse to these three circles, prove that $tt' = T^2$.

89. Find the general equation of a conic which has double contact with the two circles $(x-a)^2+y^2=c^2$, $(x-b)^2+y^2=d^2$, and prove that the equation of the locus of the extremity of the latus rectum of a conic which has double contact with the circles $(x \pm a)^2 + y^2 = c^2$ is $y^2(x^2-a^2)(x^2-a^2+c^2)=c^4x^2$.

90. Shew that the lines $lx+my=1$ and $l'x+m'y=1$ are conjugate diameters of any conic through the intersections of the two conics whose equations are

$$(l^2m' - l'^2m)x^2 + 2(l-l')mm'xy + (m-m')mm'y^2 = 2(lm'-l'm)x,$$

and

$$(m^2l' - m'^2l)y^2 + 2(m-m')ll'xy + (l-l')ll'x^2 = 2(ml'-m'l)y.$$

91. If through a fixed point chords of an ellipse be drawn, and on these as diameters circles be described, prove that the other chord of intersection of these circles with the ellipse also passes through a fixed point.

92. The angular points of a triangle are joined to two fixed points; shew that the six points, in which the joining lines meet the opposite sides of the triangle, lie on a conic.

93. If three sides of a quadrilateral inscribed in a conic pass through three fixed points in the same straight line, shew that the fourth side will also pass through a fixed point in that straight line.

94. Two chords of a conic PQ, $P'Q'$ intersect in a fixed point, and PP' passes through another fixed point. Shew that QQ' also passes through a fixed point, and that PQ', $P'Q$ touch a conic having double contact with the given conic.

95. A line parallel to one of the equi-conjugate diameters of an ellipse cuts the tangents at the ends of the major axis in the points P, Q, and the other tangents from P, Q to the ellipse meet in O; shew that the locus of O is a rectangular hyperbola.

96. L, M, N, R are fixed points on a rectangular hyperbola and P any other point on it, PA is perpendicular to LM and meets NR in a, PC is perpendicular to LN and meets MR in c, PB is perpendicular to LR and meets MN in b. Prove that $PA \cdot Pa = PB \cdot Pb = PC \cdot Pc$.

97. P is any point on a fixed diameter of a parabola. The normals from P meet the curve in A, B, C. The tangents parallel to PA, PB, PC intersect in A', B', C'. Shew that the ratio of the areas of the triangles ABC, $A'B'C'$ is constant.

98. A point P is taken on the diameter AB of a circle whose centre is C. On AP, BP as diameters circles are described: the locus of the centre of a circle which touches these three circles is an ellipse having C for one of its foci.

99. The straight lines from the centre and foci S, S' of a conic to any point intersect the corresponding chord of contact in V, G, G'; prove that the radical axis of the circles described on SG, $S'G'$ as diameters passes through V.

100. If the sides of a triangle ABC meet two given straight lines in a_1, a_2; b_1, b_2; c_1, c_2 respectively; and if round the quadrilaterals $b_1b_2c_1c_2$, $c_1c_2a_1a_2$, $a_1a_2b_1b_2$ conics be described; the three other common chords of these conics will each pass through an angular point of ABC, and will all meet in a point.

CHAPTER XI.

SYSTEMS OF CONICS.

204. THE most general equation of a conic, viz.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

contains the six constants a, h, b, g, f, c. But, since we may multiply or divide the equation by any constant quantity without changing the relation between x and y which it indicates, there are really only five constants which are fixed for any particular conic, viz. the five ratios of the six constants a, h, b, g, f, c to one another.

A conic therefore can be made to satisfy *five* conditions and no more. For example a conic can be made to pass through five given points, or to pass through four given points and to touch a given straight line. The five conditions which the conic has to satisfy give rise to five equations between the constants, and five independent equations are both necessary and sufficient to determine the five ratios.

The given equations may however give more than one set of values of the ratios, and therefore more than one conic may satisfy the given conditions; but the number of such conics will be *finite* if the conditions are really independent.

If there are only four (or less than four) conditions given, an *infinite* number of conics will satisfy them.

The five conditions which any conic can satisfy must be such that each gives rise to *one* relation among the constants; as, for instance, the condition of passing through a given point, or that of touching a given straight line.

Some conditions give two or more relations between the constants, and any such condition must be reckoned as two or more of the *five*. We proceed to give some examples.

In order that a given point may be the centre *two* relations must be satisfied [Art. 168].

To have a focus given is equivalent to having *two* tangents given [Art. 193].

To have given that a line touches a conic at a given point is equivalent to *two* conditions, for we have two consecutive points on the curve given.

To have the *direction* of an asymptote given is equivalent to having one point (at infinity) given.

To have the *position* of an asymptote given is equivalent to *two* conditions, for two points (at infinity) are given.

To have the axes given in position is equivalent to *three* conditions.

To have the eccentricity given is in general equivalent to *one* condition, but since we have $\frac{e^4}{1-e^2}=\frac{(a-b)^2+4h^2}{ab-h^2}$ [Art. 191], if we are given that $e=0$, we must have *both* $a=b$ and $h=0$.

205. *Through five points, no four of which are in a straight line, one conic and only one can be drawn.*

If three of the points are in a straight line, the conic through the five given points must be a pair of straight lines; for no straight line can meet an ellipse, parabola, or hyperbola in *three* points. And the only pair of straight lines through the five points is the line on which the three points lie and the line joining the other two points.

If however not more than two of the points are on any straight line, take the line joining two of the points for the axis of x, and the line joining two others for the axis of y.

Let the co-ordinates of the four points referred to these axes be $h_1, 0$; $h_2, 0$; $0, k_1$; and $0, k_2$ respectively.

The pairs of straight lines $\left(\frac{x}{h_1}+\frac{y}{k_1}-1\right)\left(\frac{x}{h_2}+\frac{y}{k_2}-1\right)=0$ and $xy=0$ are conics which pass through the four points.

Hence [Art. 187] all the conics given by the equation

$$\lambda xy+\left(\frac{x}{h_1}+\frac{y}{k_1}-1\right)\left(\frac{x}{h_2}+\frac{y}{k_2}-1\right)=0$$

will pass through the four points.

This conic will go through the fifth point, whose coordinates are x', y', if λ be so chosen that

$$\lambda x'y'+\left(\frac{x'}{h_1}+\frac{y'}{k_1}-1\right)\left(\frac{x'}{h_2}+\frac{y'}{k_2}-1\right)=0.$$

There is one and only one value of λ which satisfies this last equation, and therefore one and only one conic will pass through the five points.

If four points lie on a straight line, more than one conic will go through the five given points, for the straight line on which the four points lie and *any* straight line through the fifth is such a conic.

Ex. 1. Find the equation of the conic passing through the five points

$$(2,\ 1)\ (1,\ 0),\ (3,\ -1),\ (-1,\ 0) \text{ and } (3,\ -2).$$

The pairs of lines $(x-y-1)(x+4y+1)=0$, and $y(2x+y-5)=0$, pass through the first four points, and therefore also the conic

$$(x-y-1)(x+4y+1)-\lambda y(2x+y-5)=0.$$

The point $(3,\ -2)$ is on the latter conic if $\lambda=-8$; therefore the required equation is $$x^2+19xy+4y^2-45y-1=0.$$

Ex. 2. Find the equation of the conic which passes through the five points $(0,\ 0)$, $(2,\ 3)$, $(0,\ 3)$, $(2,\ 5)$ and $(4,\ 5)$.

Ans. $5x^2-10xy+4y^2+20x-12y=0.$

206. *To find the general equation of a conic through four fixed points.*

Take the line joining two of the points for axis of x, and the line joining the other two for axis of y, and let the lines whose equations are $ax+by-1=0$ and $a'x+b'y-1=0$ cut the axes in the four given points.

Then $xy=0$, and $(ax+by-1)(a'x+b'y-1)=0$ are two conics through the four points, and therefore all the conics of the system are included in the equation

$$\lambda xy + (ax + by - 1)(a'x + b'y - 1) = 0 \ldots\ldots (\text{i}),$$

or

$$aa'x^2 + (ba' + ab' + \lambda)\, xy + bb'y^2$$
$$- (a + a')\, x - (b + b')\, y + 1 = 0 \ldots\ldots\ldots (\text{ii}).$$

207. The equation (ii), Art. 206, will represent a *parabola*, if the terms of the second degree are a perfect square; that is, if

$$4aa'bb' = (ba' + ab' + \lambda)^2.$$

This equation has *two* roots, therefore *two* parabolas will pass through four given points. These parabolas are real if the roots of the equation are real, which is the case when $aa'bb'$ is positive. It is easy to shew that when $aa'bb'$ is negative the quadrilateral is re-entrant; in that case the parabolas are imaginary, as is geometrically obvious.

When the terms of the second degree in (ii), Art. 206, form a perfect square, the square must be $(\sqrt{aa'}x \pm \sqrt{bb'}y)^2$. Hence [Art. 172], the axes of the two parabolas are parallel to the lines whose equations are $\sqrt{aa'}x \pm \sqrt{bb'}y = 0$, or as one equation $aa'x^2 - bb'y^2 = 0$.

These two straight lines are parallel to conjugate diameters of any conic through the four points [Art. 183].

Hence all conics through four given points have a pair of conjugate diameters parallel to the axes of the two parabolas through those points.

208. *To find the locus of the centres of the conics which pass through four fixed points.*

As in Art. 206, the equation of any conic of the system is

$$\lambda xy + (ax + by - 1)(a'x + b'y - 1) = 0.$$

The co-ordinates of the centre of the conic are given by the equations

$$\lambda y + a(a'x + b'y - 1) + a'(ax + by - 1) = 0,$$

and

$$\lambda x + b(a'x + b'y - 1) + b'(ax + by - 1) = 0.$$

Multiply these by x and y respectively and subtract; then we have, for all values of λ,

$$(ax-by)(a'x+b'y-1)+(a'x-b'y)(ax+by-1)=0,$$

or
$$2aa'x^2-2bb'y^2-(a+a')x+(b+b')y=0.$$

The locus of the centre is therefore a conic whose asymptotes are parallel to the lines $aa'x^2-bb'y^2=0$, i.e. parallel to the axes of the two parabolas through the four points. [The two parabolas are conics of the system, and their centres are therefore the points at infinity on the centre-locus.]

209. The centre-locus in Art. 208 goes through the origin, that is through the point of intersection of the line joining two of the points and of the line joining the other two; and by symmetry it must go through the intersection of the other pairs of lines through the four points. [This could have been seen at once, for the pairs of lines are conics of the system and their centres are their points of intersection, and therefore these points of intersection are points on the centre-locus.]

The centre-locus cuts the axis of x where $x=0$ and where $x=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{a'}\right)$. Therefore the locus passes through the point midway between $\left(\frac{1}{a}, 0\right)$ and $\left(\frac{1}{a'}, 0\right)$, that is through the middle point of the line joining two of the fixed points, and therefore similarly through the middle point of the line joining any other two of the four points.

If then A, B, C, D be any four points, the three points of intersection of AB and CD, of AC and BD, and of AD and BC, together with the six middle points of AB, BC, CA, AD, BD and CD all lie on a conic, and this conic is the locus of the centres of the conics which pass through the four points A, B, C, D.

210. If aa' and bb' have the same sign, we see from Art. 208 that the centre-locus is an hyperbola, and that if aa' and bb' have different signs the centre-locus is an ellipse.

If $aa' = bb'$, that is if the four points are on a circle, the centre-locus is a rectangular hyperbola. If $aa' = -bb'$, and the axes are at right angles, all the conics of the system are rectangular hyperbolas, and the centre-locus is a circle. In this case the lines joining any two of the points is perpendicular to the line joining the other two, so that D is the ortho-centre of the triangle ABC.

Hence a circle will pass through the feet of perpendiculars of a triangle ABC and through the middle points of AB, BC, CA, AD, BD, CD where D is the ortho-centre of the triangle ABC, and this circle is the locus of the centres of all the conics (which are all rectangular hyperbolas) through A, B, C, D. This circle is called the *nine-point circle.*

211. The asymptotes of any conic through the four points defined as in Art. 206, are parallel to the lines

$$\lambda xy + (ax + by)(a'x + b'y) = 0,$$

or

$$aa'x^2 + (\lambda + ab' + a'b)xy + bb'y^2 = 0.$$

And [Art. 183] these lines are parallel to conjugate diameters of the centre-locus. Hence the asymptotes of any conic through the four points are parallel to conjugate diameters of the centre-locus; as a particular case, the asymptotes of the rectangular hyperbola which passes through the four points are parallel to the axes of the centre-locus.

Ex. 1. The polar of a fixed point with respect to a system of conics through four given points will pass through a fixed point.

Take the fixed point for origin, and let

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

and

$$S' \equiv a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0,$$

be two of the conics; then any conic of the system is given by $S - \lambda S' = 0$. The polar of the origin is

$$gx + fy + c - \lambda(g'x + f'y + c') = 0,$$

and this, for all values of λ, passes through the intersection of

$$gx + fy + c = 0, \text{ and } g'x + f'y + c' = 0.$$

Ex. 2. The locus of the poles of a given straight line with respect to the conics which pass through four given points is a conic.

Take the fixed straight line for the axis of x, and let the equation of any conic of the system be as in Ex. 1. The polar of (x', y') is

$$x(ax'+hy'+g)+y(hx'+by'+f)+gx'+fy'+c$$
$$-\lambda\{x(a'x'+h'y'+g')+y(h'x'+b'y'+f')+g'x'+f'y'+c'\}=0.$$

If this is the same line as $y=0$, the coefficient of x and the constant term must be zero. Equate these to zero and eliminate λ.

Ex. 3. Shew that the locus of the pole of a given straight line with respect to any conic which passes through the angular points of a given square is a rectangular hyperbola.

[Take for axes the lines through the centre of the square parallel to the sides; then the conics are given by $x^2-a^2-\lambda(y^2-a^2)=0$.]

Ex. 4. The nine-point circles of the four triangles determined by four given points meet in a point.

212. If $\alpha=0$ and $\beta=0$ are the equations of one pair of straight lines through four given points, and $\gamma=0$, $\delta=0$ the equations of another pair, any conic through the four points has an equation of the form

$$\alpha\beta=k\gamma\delta.$$

Now, if $\alpha=0$ be the equation of a straight line and the co-ordinates of any point be substituted in α, the result is proportional to the perpendicular distance of the point from the line. Hence the geometrical meaning of the above equation is

$$p_1p_2 \propto p_3p_4,$$

where p_1, p_2, p_3, p_4 are the perpendiculars on the four lines $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$ respectively, the perpendiculars being drawn from any point on the conic.

213. *If P, Q, R, S, be four points on a conic, and QP, RS meet in A, QS, PR in B, and PS, QR in C; then of the three points A, B, C each is the pole with respect to the conic of the line joining the other two.*

Take A for origin, and the two lines ASR, APQ for axes of x and y respectively.

Let the equations of PS and QR be

$$ax+by-1=0 \quad\text{.....................(i)},$$
$$a'x+b'y-1=0 \quad\text{.....................(ii)}.$$

Then the equations of PR and QS will be

$$a'x + by - 1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)},$$

and

$$ax + b'y - 1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

The equation of any conic through the intersection of the conics $xy = 0$ and $(ax + by - 1)(a'x + b'y - 1) = 0$, will be

$$\lambda xy + (ax + by - 1)(a'x + b'y - 1) = 0.$$

The polar of the origin of this conic is [Art. 179]

$$(a + a')x + (b + b')y - 2 = 0.$$

Writing this in the forms

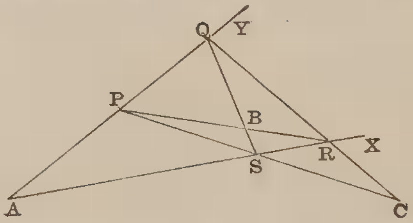
$$ax + by - 1 + a'x + b'y - 1 = 0,$$

and

$$a'x + by - 1 + ax + b'y - 1 = 0,$$

we see that the polar of the origin goes through the point of intersection of the lines (i) and (ii), and also through the point of intersection of the lines (iii) and (iv). The polar of A with respect to the conic is therefore the line BC.

It can be shewn in a similar manner that CA is the polar of B, and AB the polar of C.

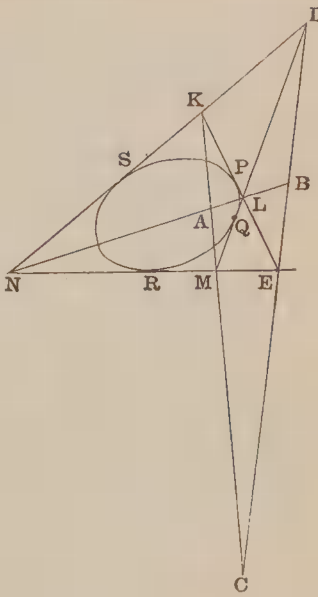


A triangle which is such that each of its angular points is the pole, with respect to a conic, of the opposite side, is called a *self-conjugate*, or *self-polar* triangle.

214. *If a conic touch the sides of a quadrilateral and ABC be the triangle formed by the diagonals of the quadrilateral; then will ABC be a self-polar triangle with respect to the conic.*

Let P, Q, R, S be the points of contact.

Then, in the figure, L is the pole of PQ, and N is the pole of SR; therefore LN is the polar of the point of intersection of PQ and SR. Similarly KM is the polar of the point of intersection of SP and RQ.



Hence A, the point of intersection of LN and KM, is the pole of the line joining the point of intersection of PQ, SR and the point of intersection of SP, RQ.

But [Art. 213] the point of intersection of PR and SQ is the pole of this last line.

Hence A is the point of intersection of PR and SQ.

So also B is the point of intersection of SP and RQ, and C is the point of intersection of PQ and SR.

Hence from Art. 213 the triangle ABC is self-polar.

215. *To find the general equation of a conic which touches the axes of co-ordinates.*

If the equation of the line joining the points of contact be $ax+by-1=0$, the equation of a conic having double contact with the conic $xy=0$, where it is met by the line $ax+by-1=0$, is [Art. 187]

$$(ax+by-1)^2-2\lambda xy=0.$$

216. *To find the general equation of a conic which touches four fixed straight lines.*

Take two of the lines for axes, and let the equations of the other two be $lx+my-1=0$, and $l'x+m'y-1=0$. The equation of any conic touching the axes is

$$(ax+by-1)^2-2\lambda xy=0 \ldots\ldots\ldots\ldots(\text{i}).$$

The lines joining the origin to the points where

$lx+my=1$ cuts (i) are given by the equation

$$(ax+by-lx-my)^2=2\lambda xy \text{ ....(ii).}$$

The line will touch the conic if the lines (ii) are coincident, the condition for which is

$$(a-l)^2(b-m)^2=\{(a-l)(b-m)-\lambda\}^2;$$

whence $$\lambda=2(a-l)(b-m).$$

Hence the general equation of a conic touching the four straight lines

$$x=0,\ y=0,\ lx+my-1=0, \text{ and } l'x+m'y-1=0,$$

is $$(ax+by-1)^2=2\lambda xy;$$

the parameters a, b, λ being connected by the two equations

$$\lambda=2(a-l)(b-m)=2(a-l')(b-m').$$

217. *To find the locus of the centres of conics which touch four given straight lines.*

If two of the lines be taken for axes, and the equations of the other two lines be

$$lx+my-1=0, \text{ and } l'x+m'y-1=0,$$

the equation of the conic will be

$$(ax+by-1)^2-2\lambda xy=0,$$

with the conditions

$$\lambda=2(a-l)(b-m) \text{ ....(i),}$$

$$\lambda=2(a-l')(b-m') \text{...............(ii).}$$

The centre of the conic is given by the equations

$$a(ax+by-1)-\lambda y=0, \text{ and } b(ax+by-1)-\lambda x=0;$$

$$\therefore\ ax=by, \text{ and } a(2ax-1)=\lambda y \text{ ....(iii).}$$

To obtain the required locus we must eliminate a, b and λ from the equations (i), (ii), and (iii).

From (i) and (iii), we have

$$a(2ax-1)=2y(a-l)(b-m)=2(a-l)(by-my);$$

therefore, since $ax=by$,

$$a(2lx+2my-1)=2lmy.$$

Similarly, from (ii) and (iii) we have

$$a(2l'x+2m'y-1)=2l'm'y.$$

Eliminating a, we obtain the equation of the locus of centres, viz.

$$\frac{2lx+2my-1}{lm}=\frac{2l'x+2m'y-1}{l'm'}.$$

The required locus is therefore the straight line whose equation is

$$2x\left(\frac{1}{m}-\frac{1}{m'}\right)+2y\left(\frac{1}{l}-\frac{1}{l'}\right)-\frac{1}{lm}+\frac{1}{l'm'}=0.$$

This straight line can easily be shewn to pass through the middle points of the diagonals of the quadrilateral, as it clearly should do, for any one of the diagonals is the limiting form of a very thin ellipse which touches the four lines, and the centre of this ellipse is ultimately the middle point of the diagonal. Hence the middle points of the three diagonals of a quadrilateral are points on the centre-locus of the conics touching the sides of the quadrilateral.

218. All conics touching the axes at the two points where they are cut by the line $ax+by-1=0$ are given by the equation

$$(ax+by-1)^2=2\lambda xy.$$

The conic will be a parabola if λ be such that the terms of the second degree form a perfect square: the condition for this is

$$a^2b^2=(ab-\lambda)^2;$$

$$\therefore\ \lambda=0,\ \text{or}\ \lambda=2ab.$$

The value $\lambda=0$ gives a pair of coincident straight lines, viz. $(ax+by-1)^2=0$.

Hence, for the parabola, $\lambda=2ab$, and the equation of the curve is

$$(ax+by-1)^2=4abxy.$$

The above equation can be reduced to the form

$$\sqrt{ax}+\sqrt{by}=1.$$

219. *To find the equation of the tangent at any point of the parabola* $\sqrt{ax}+\sqrt{by}=1$.

We may rationalize the equation of the curve and then make use of the formula obtained in Art. 177. The result may however be obtained in a simpler form as follows.

The equation of the line joining two points (x', y') and (x'', y'') on the curve is

$$\frac{x-x'}{x''-x'} = \frac{y-y'}{y''-y'}, \text{ ....(i)},$$

with the conditions

$$\sqrt{ax'} + \sqrt{by'} = 1 = \sqrt{ax''} + \sqrt{by''} \text{.........(ii)}.$$

From (ii) we have

$$\sqrt{a}\,(\sqrt{x'} - \sqrt{x''}) = -\sqrt{b}\,(\sqrt{y'} - \sqrt{y''}) \text{ ....(iii)}.$$

Multiply the corresponding sides of the equations (i) and (iii), and we have

$$\frac{\sqrt{a}}{\sqrt{x'}+\sqrt{x''}}(x-x') = -\frac{\sqrt{b}}{\sqrt{y'}+\sqrt{y''}}(y-y').$$

The equation of the tangent at (x', y') is therefore

$$\frac{\sqrt{a}}{\sqrt{x'}}(x-x') + \frac{\sqrt{b}}{\sqrt{y'}}(y-y') = 0,$$

or, since $\sqrt{ax'} + \sqrt{by'} = 1$,

$$x\sqrt{\frac{a}{x'}} + y\sqrt{\frac{b}{y'}} = 1.$$

To find the equation of the polar of any point with respect to the conic, we must use the rationalized form of the equation of the parabola.

Ex. 1. *To find the condition that the line* $lx + my - 1 = 0$ *may touch the parabola* $\sqrt{ax} + \sqrt{by} - 1 = 0$.

The equation of the tangent at any point (x', y') is

$$x\sqrt{\frac{a}{x'}} + y\sqrt{\frac{b}{y'}} = 1;$$

which is the same as the given equation, if $l = \sqrt{\frac{a}{x'}}$, and $m = \sqrt{\frac{b}{y'}}$;

or if $\frac{a}{l} = \sqrt{ax'}$, and $\frac{b}{m} = \sqrt{by'}$.

Hence the required condition is

$$\frac{a}{l} + \frac{b}{m} = 1.$$

Ex. 2. *To find the focus of the parabola whose equation is*

$$\sqrt{ax}+\sqrt{by}=1.$$

The circle which touches TQ at T and which passes through P will also pass through the focus [see Art. 165 (4), two of the tangents being coincident]. The two points P, Q are $\left(\frac{1}{a},\ 0\right)$ and $\left(0,\ \frac{1}{b}\right)$. Therefore the focus is on both the circles whose equations are

$$x^2+2xy\cos\omega+y^2-\frac{x}{a}=0,$$

and

$$x^2+2xy\cos\omega+y^2-\frac{y}{b}=0.$$

Hence the focus is given by

$$x^2+y^2+2xy\cos\omega=\frac{x}{a}=\frac{y}{b}.$$

Ex. 3. *To find the directrix of the parabola* $\sqrt{ax}+\sqrt{by}=1$.

The directrix is the locus of the intersection of tangents at right angles; now the line $lx+my=1$ will be perpendicular to $y=0$ if $m-l\cos\omega=0$, and the line will touch if $\frac{a}{l}+\frac{b}{m}=1$. Therefore the intercept on the axis of x made by a tangent perpendicular to that axis is given by $\frac{1}{l}\left(a+\frac{b}{\cos\omega}\right)=1$.

Hence the point $\left(\frac{\cos\omega}{b+a\cos\omega},\ 0\right)$ is on the directrix.

Similarly the point $\left(0,\ \frac{\cos\omega}{a+b\cos\omega}\right)$ is on the directrix.

Hence the required equation is

$$x(b+a\cos\omega)+y(a+b\cos\omega)=\cos\omega.$$

220. Since the foci of a conic are on its axes, if two conics are confocal they must have the same axes.

The equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

will, for different values of λ, represent different conics of a confocal system. For the distance of a focus from the centre is

$$\sqrt{\{(a^2+\lambda)-(b^2+\lambda)\}} \text{ or } \sqrt{\{a^2-b^2\}}.$$

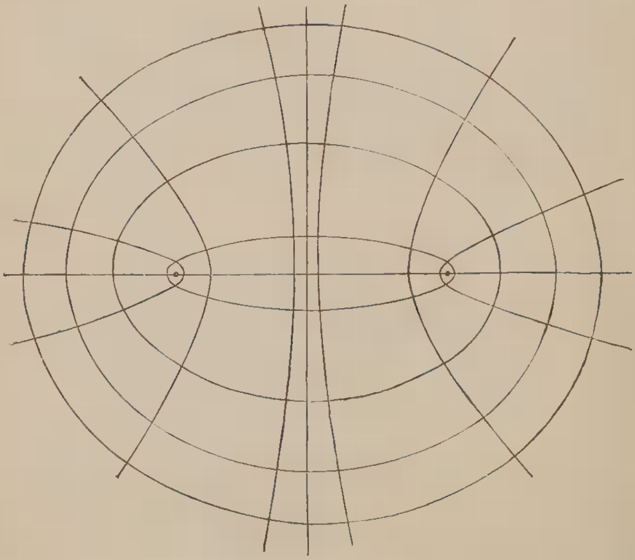
221. The equation of a system of confocal conics is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

If λ is positive the curve is an ellipse.

The principal axes of the curve will increase as λ increases, and their ratio will tend more and more to equality as λ is increased more and more; so that a circle of infinite radius is a limiting form of one of the confocals.

If λ be negative, the principal axes will decrease as λ increases, and the ratio $\dfrac{b^2+\lambda}{a^2+\lambda}$ will also decrease as λ increases, so that the ellipse becomes flatter and flatter, until λ is equal to $-b^2$, when the minor axis vanishes, and the major axis is equal to the distance between the foci. Hence the line-ellipse joining the foci is a limiting form of one of the confocals.



If $b^2+\lambda$ is negative, the curve is an hyperbola.

If $b^2+\lambda$ is a *small* negative quantity the transverse axis of the hyperbola is very nearly equal to the distance between the foci; and the *complement* of the line joining the foci is a limiting form of the hyperbola.

The angle between the asymptotes of the hyperbola will become greater and greater as $-\lambda$ becomes greater and greater and in the limit both branches of the curve coincide with the axis of y.

If λ is negative and numerically greater than a^2, the curve is imaginary.

222. *Two conics of a confocal system pass through any given point. One of these conics is an ellipse and the other an hyperbola.*

Let the equation of the original conic be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The equation of any confocal conic is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

This will pass through the given point (x', y'), if

$$\frac{x'^2}{a^2+\lambda}+\frac{y'^2}{b^2+\lambda}=1.$$

In the above put $b^2+\lambda=\lambda'$;

then $$x'^2\lambda'+y'^2(\lambda'+a^2e^2)-\lambda'(\lambda'+a^2e^2)=0,$$

or $$\lambda'^2-\lambda'(x'^2+y'^2-a^2e^2)-a^2e^2y'^2=0.$$

The roots of this quadratic in λ' are both real, and are of different signs. Therefore there are *two* conics, and $b^2+\lambda$ is positive for one, and negative for the other, so that one conic is an ellipse and the other an hyperbola.

223. *One conic of a confocal system and only one will touch a given straight line.*

Let the equation of the given straight line be

$$lx+my-1=0.$$

The line will touch the conic whose equation is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

if $$(a^2+\lambda)\,l^2+(b^2+\lambda)\,m^2=1 \quad \text{[Art. 115]},$$

which gives one, and only one, value of λ. Hence one confocal will touch the given straight line.

224. *Two confocal conics cut one another at right angles at all their common points.*

Let the equations of the conics be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ and } \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

and let (x', y') be a common point; then the co-ordinates x', y' will satisfy both the above equations.

Hence, by subtraction, we have

$$\frac{x'^2}{a^2(a^2+\lambda)}+\frac{y'^2}{b^2(b^2+\lambda)}=0 \ldots\ldots\ldots \text{(i)}.$$

Now the equations of the tangents to the conics at (x', y') are

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1, \text{ and } \frac{xx'}{a^2+\lambda}+\frac{yy'}{b^2+\lambda}=1$$

respectively.

The condition (i) shews that the tangents are at right angles to one another.

225. *The difference of the squares of the perpendiculars drawn from the centre on any two parallel tangents to two given confocal conics is constant.*

Let the equations of the conics be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ and } \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

Let the two straight lines

$$x\cos\alpha+y\sin\alpha-p=0, \quad x\cos\alpha+y\sin\alpha-p'=0,$$

touch the conics respectively; then [Art. 115, Cor.] we have

$$p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha,$$

and

$$p'^2 = (a^2 + \lambda) \cos^2 \alpha + (b^2 + \lambda) \sin^2 \alpha;$$

$$\therefore\ p'^2 - p^2 = \lambda.$$

226. *If a tangent to one of two confocal conics be perpendicular to a tangent to the other, the locus of their point of intersection is a circle.*

Let the equations of the confocal conics be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ and } \frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1.$$

The lines whose equations are

$$x \cos \alpha + y \sin \alpha = \sqrt{(a^2 \cos^2 \alpha + b^2 \sin^2 \alpha)} \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

$$x \sin \alpha - y \cos \alpha = \sqrt{\{(a^2 + \lambda) \sin^2 \alpha + (b^2 + \lambda) \cos^2 \alpha\}} \ldots\text{(ii)},$$

touch the conics respectively, and are at right angles to one another.

Square both sides of the equations (i) and (ii) and add, then we have for the equation of the required locus

$$x^2 + y^2 = a^2 + b^2 + \lambda.$$

If we suppose the minor axis of the second ellipse to become indefinitely small, all tangents to it will pass indefinitely near to a focus; so that Art. 125 (η) is a particular case of the above.

Ex. 1. Any two parabolas which have a common focus and their axes in opposite directions intersect at right angles.

Ex. 2. Two parabolas have a common focus and their axes in the same straight line; shew that, if TP, TQ be tangents one to each of the parabolas, and TP, TQ be at right angles to one another, the locus of T is a straight line.

Ex. 3. TQ, TP are tangents one to each of two confocal conics whose centre is C; shew that if the tangents are at right angles to one another CT will bisect PQ.

Let the tangents be

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1, \text{ and } \frac{xx''}{a'^2} + \frac{yy''}{b'^2} = 1,$$

the equation of CT will be

$$x\left(\frac{x'}{a^2}-\frac{x''}{a'^2}\right)+y\left(\frac{y'}{b^2}-\frac{y''}{b'^2}\right)=0.$$

This will pass through the middle point of PQ, if

$$(x'+x'')\left(\frac{x'}{a^2}-\frac{x''}{a'^2}\right)+(y'+y'')\left(\frac{y'}{b^2}-\frac{y''}{b'^2}\right)=0;$$

that is, if

$$x'x''\left(\frac{1}{a^2}-\frac{1}{a'^2}\right)+y'y''\left(\frac{1}{b^2}-\frac{1}{b'^2}\right)=0;$$

or, since the conics are confocal, if

$$\frac{x'x''}{a^2a'^2}+\frac{y'y''}{b^2b'^2}=0.$$

That is, if the tangents are at right angles.

Ex. 4. TP, TQ are tangents one to each of two parabolas which have a common focus and their axes in the same straight line; shew that, if a line through T parallel to the axis bisect PQ, the tangents will be at right angles.

Ex. 5. If points on two confocal ellipses which have the same eccentric angles are called corresponding points; shew that, if P,Q be any two points on an ellipse, and p, q be the corresponding points on a confocal ellipse, then $Pq=Qp$.

227. *The locus of the pole of a given straight line with respect to a series of confocal conics is a straight line.*

Let the equation of the confocals be

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

and let the equation of the given straight line be

$$lx+my=1\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

The equation of the polar of the point (x', y') with respect to (i) is $\quad \dfrac{xx'}{a^2+\lambda}+\dfrac{yy'}{b^2+\lambda}=1\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$

If (ii) and (iii) represent the same straight line, we must have $\quad \dfrac{x'}{a^2+\lambda}=l,\quad \dfrac{y'}{b^2+\lambda}=m;$

$$\therefore\quad \frac{x'}{l}-a^2=\frac{y'}{m}-b^2=\lambda.$$

Hence the locus of the poles is the straight line whose equation is

$$\frac{x}{l}-\frac{y}{m}=a^2-b^2.$$

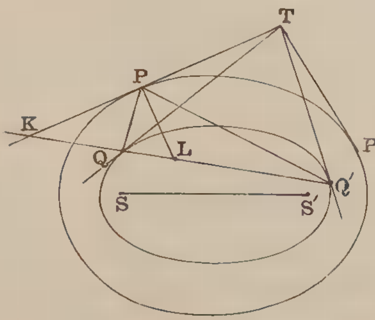
This straight line is perpendicular to the line (ii). One confocal of the system will touch the line (ii), and the point of contact will be the pole of the line with respect to that confocal.

Hence the locus of the poles is a straight line perpendicular to the given straight line and through the point where it touches a confocal.

228. *From any point T the two tangents TP, TP' are drawn to one conic, and the two tangents TQ, TQ' to a confocal conic; shew that the straight lines QP, $Q'P$ will make equal angles with the tangent at P.*

Let TP and the normal at P cut QQ' in K, L respectively.

Then [Art. 227] the pole of TP, with respect to the conic on which Q, Q' lie, is on the line PL. Also, since T is the pole of QQ' with respect to that conic, the pole of TP is on QQ' [Art. 180]. Therefore the pole of TPK is at L, the point of intersection of QQ' and PL.



Therefore [Art. 181] the range K, Q, L, Q', and the pencil PK, PQ, PL, PQ', are harmonic.

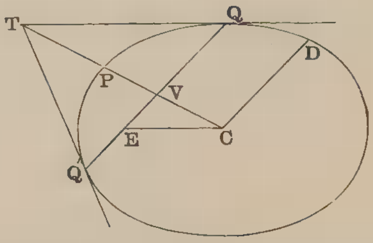
Hence, since the angle KPL is a right angle, PQ and PQ' make equal angles with PL or PK [Art. 56].

Cor. 1. Let the conic on which Q, Q' lie degenerate into the line-ellipse joining the foci, then the proposition becomes—*The lines joining the foci of a conic to any point P on the curve make equal angles with the tangent at P.*

Cor. 2. Let the conic on which P, P' lie degenerate into the line-ellipse, and we have—*Two tangents to a conic subtend equal angles at a focus.*

Cor. 3. Let the conic on which P, P' lie pass through T, and we have—*The two tangents drawn to a conic from any point T make equal angles with the tangent at T to either of the confocal conics which pass through T.*

229. *If QQ' be any chord of a given conic which touches a fixed confocal conic, then will QQ' vary as the square of the parallel diameter. Also, if CE be drawn through the centre parallel to the tangent at Q and meeting QQ' in E, then will QE be of constant length.*



Let T be the pole of QQ', and let CT cut QQ' in V, and the curve in P. Also let CD be the semi-diameter parallel to QQ'.

Let p', p be the lengths of the perpendiculars from the centre on QQ', and on the parallel tangent to the ellipse QPQ'; then [Art. 225] we have

$$p^2 - p'^2 = \lambda.$$

Hence $$\frac{\lambda}{p^2} = 1 - \frac{p'^2}{p^2} = 1 - \frac{CV^2}{CP^2} = \frac{QV^2}{CD^2};$$
therefore, since $p \,.\, CD = ab$, we have
$$QV^2 = \lambda \,.\, \frac{CD^2}{p^2} = \frac{\lambda}{a^2b^2} \,.\, CD^4;$$
$$\therefore QV = \frac{\sqrt{\lambda}}{ab}\, CD^2 \text{........................(i).}$$

Also
$$\frac{QE}{QV} = \frac{CT}{VT} = \frac{CV \,.\, CT}{CV \,.\, CT - CV^2} = \frac{CP^2}{CP^2 - CV^2} = \frac{CD^2}{QV^2}$$
therefore from (i) we have
$$QE = \frac{CD^2}{QV} = \frac{ab}{\sqrt{\lambda}} \text{..............(ii).}$$

Ex. *TP, TQ are tangents one to each of two fixed confocal conics; shew that, if the tangents are at right angles to one another, the line PQ will always touch a third confocal conic.*

If C be the common centre, then since the tangents are at right angles to one another the line CT bisects PQ [Ex. (3) Art. 226]. Therefore CT and QP make equal angles with the tangent at Q. If therefore CE be parallel to the tangent at Q, and meet QP in E, we have $QE = CT$.

But CT is constant [Art. 226]. Hence QE is constant, and therefore QEP touches a fixed confocal.

230. When two of the points of intersection of any two curves are coincident, that is when the two curves touch, they are said to have *contact of the first order* at the point. When *three* points of intersection are coincident the curves are said to have contact of the *second order*, and so on.

A curve which has with a given curve a contact of the highest possible order is called an *osculating* curve.

A circle can only be made to pass through *three* given points; hence the circles which osculate a curve have contact of the second order with it.

The circle which has contact of the second order with a given curve at a given point is generally called the *circle* *of curvature* at that point, and the radius of the circle is called the *radius of curvature* at the point.

Two conics intersect in four points. Hence two conics cannot have contact with one another of higher order than the third. If they have contact of the second order they will have *one* other common point.

231. *To find the general equation of a conic which has contact of the second order with a given conic at a given point.*

Let $S=0$ be the equation of the given conic, and let $T=0$ be the equation of the tangent to $S=0$ at the given point (x', y').

The equation of any straight line through (x', y') is

$$y-y'-m(x-x')=0.$$

Hence the equation

$$S-\lambda T\{(y-y')-m(x-x')\}=0 \;\ldots\ldots\text{(i)}$$

is the equation of a conic passing through the points where the straight lines $T=0$, and $y-y'-m(x-x')=0$ cut $S=0$.

Hence (i) intersects $S=0$ in *three coincident* points.

The two constants λ and m being arbitrary, the conic given by (i) can be made to satisfy two other conditions. They can for instance be so chosen that the equation (i) shall represent a circle.

If the line $y-y'-m(x-x')=0$ coincides with the tangent, all four points of intersection are coincident. The conic $S-\lambda T^2=0$ therefore has contact of the *third* order with $S=0$; that is to say, is the equation of an *osculating* conic.

Ex. 1. Find the equation of the circle which osculates the conic $ax^2+2bxy+cy^2+2dx=0$ at the origin.

All the conics included in the equation

$$ax^2+2bxy+cy^2+2dx-\lambda x(y-mx)=0$$

have contact of the second order.

The conditions for a circle are $2b-\lambda=0$ and $a+\lambda m=c$. Therefore the circle required is $cx^2+cy^2+2dx=0$.

Ex. 2. Find the equation of the parabola which has contact of the third order with the conic $ax^2+2bxy+cy^2+2dx=0$ at the origin.

The conic $ax^2+2bxy+cy^2+2dx-\lambda x^2=0$ cuts the given conic in four coincident points.

The curve is a parabola if $(a-\lambda)c=b^2$.

The equation of the required parabola is therefore

$$b^2x^2+2bcxy+c^2y^2+2dcx=0.$$

232. Since the line joining any two of the points of intersection of a circle and a conic, and the line joining the other two points of intersection, make equal angles with the axis of the conic, we see that, if the circle of curvature at a point P of a conic cut the conic again in O, the tangent at P and the chord PO make equal angles with an axis of the conic.

233. If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of four points on an ellipse, a circle will pass through those four points, if

$$\alpha+\beta+\gamma+\delta=2n\pi \text{ [Art. 184. Ex. 1].}$$

Hence the circle of curvature at the point α will cut the ellipse again at the point δ where

$$3\alpha+\delta=2n\pi \text{.....................(i).}$$

From (i) we see that, through any particular point δ three circles of curvature will pass, viz. the circles of curvature at the points $\frac{1}{3}(2\pi-\delta)$, $\frac{1}{3}(4\pi-\delta)$, and $\frac{1}{3}(6\pi-\delta)$. These three points are the angular points of a maximum triangle inscribed in the ellipse [Art. 138 (1)]. Also, since $\delta+\frac{1}{3}(2\pi-\delta)+\frac{1}{3}(4\pi-\delta)+\frac{1}{3}(6\pi-\delta)=4\pi$, the point δ and the three points the circles of curvature at which pass through δ are on a circle.

234. *To find the equations of the three pairs of straight lines which can be drawn through the points of intersection of two conics.*

Let the equations of the conics be

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0,$$

and $$S'\equiv a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0.$$

The equation of any conic through their points of intersection is of the form

$$S + \lambda S' = 0 \text{ ....(i).}$$

The conic $S + \lambda S' = 0$ will be a pair of straight lines, if

$$\begin{vmatrix} a + \lambda a', & h + \lambda h', & g + \lambda g' \\ h + \lambda h', & b + \lambda b', & f + \lambda f' \\ g + \lambda g', & f + \lambda f', & c + \lambda c' \end{vmatrix} = 0 \text{ ......(ii).}$$

We have therefore a cubic for the determination of λ. If any root of this cubic be substituted in (i) we have the equation of one of the three pairs of straight lines.

If λ be eliminated between the equations (i) and (ii) we have an equation of the sixth degree which represents the three pairs of straight lines.

Since one root of a cubic equation is always real, one value of λ is in all cases real.

It can be shewn that at least one pair of straight lines is in all cases real. [See Salmon's Conics, Art. 282.]

235. The equation (ii) Art. 234 is usually written

$$\Delta + \lambda\Theta + \lambda^2\Theta' + \lambda^3\Delta' = 0.$$

If the axes be changed in any manner, and the equations of the two conics become $\Sigma = 0$ and $\Sigma' = 0$, the equation $S + \lambda S' = 0$ will become $\Sigma + \lambda\Sigma' = 0$; and if λ be such that $S + \lambda S' = 0$ represents a pair of straight lines, so also will $\Sigma + \lambda\Sigma' = 0$. Hence the values of λ for which $S + \lambda S' = 0$ represents straight lines must be independent of any particular axes of co-ordinates; hence the ratios of the four quantities Δ, Θ, Θ', Δ' to one another must be independent of the axes of co-ordinates. For this reason they are called the *Invariants* of the system. The student will find interesting applications of invariants in Salmon's Conic Sections and Wolstenholme's Problems.

236. We shall conclude this Chapter by the solution of some Examples.

Ex. 1. *If two conics have each double contact with a third, their chords of contact with that conic, and two of the lines through their common points, will meet in a point and form a harmonic pencil.*

Let $S=0$ be the equation of the third conic, and let $\alpha=0$, $\beta=0$ be the equations of the two chords of contact. Then [Art. 187] the equations of the conics are

$$S-\lambda^2\alpha^2=0 \text{ ....(i)},$$

and

$$S-\mu^2\beta^2=0 \text{ ....(ii)}.$$

Now the two straight lines

$$\lambda^2\alpha^2-\mu^2\beta^2=0 \text{ ....(iii)}$$

go through the common points of (i) and (ii). The lines (iii) also go through the point of intersection of $\alpha=0$ and $\beta=0$; and [Art. 56] the four lines $\alpha=0$, $\lambda\alpha-\mu\beta=0$, $\beta=0$, and $\lambda\alpha+\mu\beta=0$ form a harmonic pencil.

Ex. 2. *A circle of given radius cuts an ellipse in four points; shew that the continued product of the diameters of the ellipse parallel to the common chords is constant.*

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the equation of the circle be $(x-\alpha)^2+(y-\beta)^2-k^2=0$. Then the equation of any pair of common chords is

$$(x-\alpha)^2+(y-\beta)^2-k^2-\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=0 \text{ ....(i)},$$

where λ is one of the roots of the equation

$$\begin{vmatrix} 1-\frac{\lambda}{a^2}, & 0, & -\alpha \\ 0, & 1-\frac{\lambda}{b^2}, & -\beta \\ -\alpha, & -\beta, & \lambda+\alpha^2+\beta^2-k^2 \end{vmatrix}=0 \text{ ....(ii)}.$$

The equation of the diameters of the ellipse parallel to the lines (i) is

$$x^2+y^2-\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=0 \text{ ....(iii)}.$$

The two semi-diameters given by (iii) clearly make equal angles with the axis, and the square of the length of one of them is equal to λ.

Hence the continued product of the six semi-diameters is equal to the product of the three values of λ given by (ii), which is easily seen to be $a^2b^2k^2$.

Ex. 3. *If a conic have any one of four given points for centre, and the triangle formed by the other three for a self polar triangle, its asymptotes will be parallel to the axes of the two parabolas which pass through the four points.*

Let the four points be given by the intersections of the straight lines

$$xy=0 \text{ and } (lx+my-1)(l'x+m'y-1)=0.$$

The line joining the centre of a conic to any one of the angular points of a self polar triangle is conjugate to the line joining the other two angular points. Hence, for all the four conics, the three pairs of lines joining the four given points are parallel to conjugate diameters.

Let the equation of one of the conics be

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad\text{.................(i).}$$

The lines $(lx+my-1)(l'x+m'y-1)=0$

are parallel to conjugate diameters; therefore also the lines

$$ll'x^2+(lm'+l'm)\,xy+mm'y^2=0$$

are parallel to conjugate diameters. Hence [Art. 183], we have

$$amm'+bll'=h\,(lm'+l'm).$$

The lines $xy=0$ are parallel to conjugate diameters; therefore $h=0$, and we have

$$amm'+bll'=0 \quad\text{............................(ii).}$$

The asymptotes of (i) are parallel to the straight lines

$$ax^2+by^2=0,$$

or, from (ii), the asymptotes are parallel to the lines

$$ll'x^2-mm'y^2=0,$$

which proves the theorem [Art. 207].

Ex. 4. *The circumscribing circle of any triangle self polar with respect to a conic cuts the director-circle orthogonally.*

Let the equation of the conic be $ax^2+by^2=1$; and let (x', y'), (x'', y'') and (x''', y''') be the angular points of the triangle.

Since each of the points is on the polar of another, we have

$$ax''x'''+by''y'''-1=0 \quad\text{......................... (i),}$$

$$ax'''x'+by'''y'-1=0 \quad\text{.........................(ii),}$$

and

$$ax'x''+by'y''-1=0 \quad\text{...... ....(iii).}$$

The equation of the circle circumscribing the triangle is

$$\begin{vmatrix} x^2+y^2, & x, & y, & 1 \\ x'^2+y'^2, & x', & y', & 1 \\ x''^2+y''^2, & x'', & y'', & 1 \\ x'''^2+y'''^2, & x''', & y''', & 1 \end{vmatrix}=0 \quad\text{.................. (iv).}$$

Now, if the equation of a circle be

$$Ax^2+Ay^2+2Gx+2Fy+C=0,$$

the square of the tangent to it from the origin is equal to the ratio of C to A.

Hence the square of the tangent to the circle (iv) is equal to the ratio of

$$\begin{vmatrix} x'^2+y'^2, & x', & y' \\ x''^2+y''^2, & x'', & y'' \\ x'''^2+y'''^2, & x''', & y''' \end{vmatrix} \text{ to } - \begin{vmatrix} x', & y', & 1 \\ x'', & y'', & 1 \\ x''', & y''', & 1 \end{vmatrix}.$$

The first determinant is equal to

$$x'^2(x''y'''-y''x''')+x''^2(x'''y'-y'''x')+x'''^2(x'y''-y'x'')$$
$$+y'^2(x''y'''-y''x''')+y''^2(x'''y'-y'''x')+y'''^2(x'y''-y'x'')\ldots\ldots(\alpha).$$

Now from the equations (i), (ii), (iii) we have

$$\frac{ax'}{y'''-y''}=\frac{by'}{x''-x'''}=\frac{-1}{x'''y''-y'''x''},$$

$$\frac{ax''}{y'-y'''}=\frac{by''}{x'''-x'}=\frac{-1}{x'y'''-y'x'''},$$

and

$$\frac{ax'''}{y''-y'}=\frac{by'''}{x'-x''}=\frac{-1}{x''y'-y''x'}.$$

By means of these equations, (α) becomes

$$\frac{x'}{a}(y'''-y'')+\frac{x''}{a}(y'-y''')+\frac{x'''}{a}(y''-y')$$
$$+\frac{y'}{b}(x''-x''')+\frac{y''}{b}(x'''-x')+\frac{y'''}{b}(x'-x''),$$

or

$$-\left(\frac{1}{a}+\frac{1}{b}\right)\begin{vmatrix} x', & y', & 1 \\ x'', & y'', & 1 \\ x''', & y''', & 1 \end{vmatrix}.$$

Hence the tangent to the circumscribing circle from the centre of the conic is equal to $\sqrt{\left(\frac{1}{a}+\frac{1}{b}\right)}$, that is equal to the radius of the director-circle, which proves the proposition.

EXAMPLES ON CHAPTER XI.

1. Two straight lines of given length are moved along two given straight lines in such a manner that a circle will pass through their four extremities; shew that the locus of the centre of this circle is a rectangular hyperbola.

2. OPP', OQQ' are two chords of a conic, and any line through O cuts the conic in R, R' and the lines PQ, $P'Q'$ in S, S'; shew that

$$\frac{1}{OR}+\frac{1}{OR'}=\frac{1}{OS}+\frac{1}{OS'}.$$

3. A system of conics pass through the same four points and the tangent at a given point O of one of the conics cuts any other of the conics in P, P'; shew that $\frac{1}{OP} + \frac{1}{OP'}$ is constant.

4. A circle and a rectangular hyperbola intersect in four points, and one of their common chords is a diameter of the hyperbola; shew that the other chord is a diameter of the circle.

5. Of all conics which pass through four given points that which has the least eccentricity has its equi-conjugate diameters parallel to the axes of the two parabolas through the points.

6. Of all conics which touch two given straight lines at given points the one of least eccentricity will be that in which one of the equi-conjugate diameters passes through the intersection of the given lines.

7. The locus of the middle point of the intercept of a variable tangent to a conic on two fixed tangents OA, OB is a conic which reduces to a straight line if the original conic is a parabola.

8. Two tangents OA, OB are drawn to a conic and are cut in P and Q by a variable tangent; prove that the locus of the centre of the circle described about the triangle OPQ is an hyperbola.

9. A conic is drawn touching the co-ordinate axes OX, OY at A, B and passing through the point D where $OADB$ is a parallelogram; shew that if the area of the triangle OAB be constant, the locus of the centre of the conic will be an hyperbola.

10. Tangents are drawn from a fixed point to a system of conics touching two given straight lines at given points. Prove that the locus of the point of contact is a conic.

11. Shew that the locus of the pole of a given straight line with respect to a series of conics inscribed in the same quadrilateral is a straight line.

12. An ellipse is described touching the asymptotes of an hyperbola and meeting the hyperbola in four points; shew that two of the common chords are parallel to the line joining the points of contact of the ellipse with the asymptotes, and are equidistant from that line.

13. In a system of conics which have a given centre and their axes in a given direction, the sum of the axes is given; shew that the locus of the pole of a given straight line is a parabola touching the axes.

14. A parabola is drawn so as to touch three given straight lines; shew that the chords joining the points of contact pass each through a fixed point.

15. Shew that, if a parabola touch two given straight lines, and the line joining the points of contact pass through a fixed point, the locus of the focus will be a circle.

16. If the axis of the parabola $\sqrt{ax} + \sqrt{by} = 1$ pass through a fixed point, the locus of the focus will be a rectangular hyperbola.

17. From a fixed point O, a pair of secants are drawn meeting a given conic in four points lying on a circle; shew that the locus of the centre of this circle is the perpendicular from O on the polar of O.

18. TP, TQ are tangents to a conic, and R any other point on the curve; RQ, RP meet any straight line through T in the points K, L respectively; shew that QL and PK intersect on the curve.

19. Any point P on a fixed straight line is joined to two fixed points A, B of a conic, and the lines PA, PB meet the conic again in Q, R; shew that the locus of the point of intersection of BQ and AR is a conic.

20. The confocal hyperbola through the point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose eccentric angle is α has for equation

$$\frac{x^2}{\cos^2 \alpha} - \frac{y^2}{\sin^2 \alpha} = a^2 - b^2.$$

21. Find the locus of the points of contact of tangents to a series of confocal conics from a given point in the major axis.

22. If λ, μ be the parameters of the confocals which pass through two points P, Q on a given ellipse; shew (i) that if P, Q be extremities of conjugate diameters then $\lambda + \mu$ is constant, and (ii) that if the tangents at P and Q be at right angles then $\frac{1}{\lambda} + \frac{1}{\mu}$ is constant.

23. Shew that the ends of the equal conjugate diameters of a series of confocal ellipses are on a confocal rectangular hyperbola.

24. Find the angle between the two tangents to an ellipse from any point in terms of the parameters of the confocals through that point; and shew that the equation of the two tangents referred to the normals to the confocals as axes will be

$$\frac{x^2}{\lambda_1}+\frac{y^2}{\lambda_2}=0.$$

25. The straight lines OPP', OQQ' cut an ellipse in P, P' and Q, Q' respectively and touch a confocal ellipse; prove that $OP \,.\, OP' \,.\, QQ' = OQ \,.\, OQ' \,.\, PP'$.

26. The locus of the points of contact of the tangents drawn from a given point to a system of confocals is a cubic curve, which passes through the given point and through the foci.

27. Shew that the locus of the points of contact of parallel tangents to a system of confocals is a rectangular hyperbola; and the locus of the vertices of these hyperbolas for all possible directions of the tangent is the curve whose equation is

$$r^2=(a^2-b^2)\cos 2\theta.$$

28. If a triangle be inscribed in an ellipse and envelope a confocal ellipse, the points of contact will lie on the escribed circles of the triangle.

29. If an ellipse have double contact with each of two confocals, the tangents at the points of contact will form a rectangle.

30. If from a fixed point tangents be drawn to one of a given system of confocal conics, and the normals at the points of contact meet in Q, shew that the locus of Q is a straight line.

31. A triangle circumscribes an ellipse and two of its angular points lie on a confocal ellipse; prove that the third angular point lies on another confocal ellipse.

32. An ellipse and hyperbola are confocal, and the asymptotes of the hyperbola lie along the equi-conjugate diameters of the ellipses; prove that the hyperbola will cut at right angles all conics which pass through the ends of the axes of the ellipse.

33. Four normals are drawn to an ellipse from a point P; prove that their product is

$$\frac{\lambda_1\lambda_2(\lambda_1-\lambda_2)}{a^2-b^2},$$

where λ_1, λ_2 are the parameters of the confocals to the given ellipse which pass through P and a, b the semi-axes of the given ellipse.

34. Shew that the feet of the perpendiculars of a triangle are a conjugate triad with respect to any equilateral hyperbola which circumscribes the triangle.

35. TP, TQ are the tangents from a point T to a conic, and the bisector of the angle PTQ meets PQ in O; shew that, if ROR' be any other chord through O, the angle RTR' will be bisected by OT.

36. If two parabolas are drawn each passing through three points on a circle and one of them meeting the circle again in D, the other meeting it again in E; prove that the angle between their axes is one-fourth of the angle subtended by DE at the centre of the circle.

37. If ABC be a maximum triangle inscribed in an ellipse and the circle round ABC cut the ellipse again in D; shew that the locus of the point of intersection of the axes of the two parabolas which pass through A, B, C, D is a conic similar to the original conic.

38. If any point on a circle of radius a be given by the co-ordinates $a\cos\theta$, $a\sin\theta$; shew that the equations of the axes of the two parabolas through the four points α, β, γ, δ are

$$x\cos S+y\sin S=\frac{a}{4}\left\{\begin{matrix}\cos(S-\alpha)+\cos(S-\beta)+\cos(S-\gamma)\\+\cos(S-\delta)\end{matrix}\right\},$$

$$x\sin S-y\cos S=\frac{a}{4}\left\{\begin{matrix}\sin(S-\alpha)+\sin(S-\beta)+\sin(S-\gamma)\\+\sin(S-\delta)\end{matrix}\right\},$$

where $$4S=\alpha+\beta+\gamma+\delta.$$

If the axes of the two parabolas intersect in P, shew that the five points so obtained by selecting four out of five points on the circle in all possible ways, lie on a circle of radius $\frac{a}{4}$.

39. If A, B, C, D be the *sides* of a quadrilateral inscribed in a conic, the ratio of the product of the perpendiculars from any point P of the circle on the sides A and C to the product of the perpendiculars on the sides B and D will be constant. Shew also, that if A, B, C, D, E, F...be the sides of a polygon inscribed in the conic, the number of sides being even, the continued product of the perpendiculars from any point on the conic on the sides A, C, E,...will be to the continued product of the perpendiculars from the same point on the sides B, D, F,...in a constant ratio.

40. O is the centre of curvature at any point of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; Q, R are the feet of the other two normals drawn from O to the ellipse; prove that, if the tangents at Q and R meet in T, the equation of the locus of T is $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$.

41. Shew that a circle cannot cut a parabola in four real points if the abscissa of its centre be less than the semi-latus rectum.

A circle is described cutting a parabola in four points, and through the vertex of the parabola lines are drawn parallel to the six lines joining the pairs of points of intersection; shew that the sum of the abscissæ of the points where these lines cut the parabola is constant if the abscissa of the centre of the circle is constant.

42. Three straight lines form a self-polar triangle with respect to a rectangular hyperbola. The curve being supposed to vary while the lines remain fixed, find the locus of the centre.

43. If a circle be described concentric with an ellipse, shew that an infinite number of triangles can be inscribed in the ellipse and circumscribed about the circle, if $\frac{1}{c}=\frac{1}{a}+\frac{1}{b}$, where c is the radius of the circle, and a, b the semi-axes of the ellipse.

44. Find the points on an ellipse such that the osculating circle at P passes through Q, and the osculating circle at Q passes through P.

45. Prove that the locus of the centres of rectangular hyperbolas which have contact of the third order with a given parabola is an equal parabola.

46. P, Q are two points on an ellipse: prove that if the normal at P bisects the angle that the normal at Q subtends at P, the normal at Q will bisect the angle the normal at P subtends at Q.

47. Shew that the centre of curvature at any point P of an ellipse is the pole of the tangent at P with respect to the confocal hyperbola through P.

48. ABC is a triangle inscribed in an ellipse. A confocal ellipse touches the sides in A', B', C'. Prove that the confocal hyperbola through A meets the inner ellipse in A'.

49. Of two rectangular hyperbolas the asymptotes of one are parallel to the axes of the other and the centre of each lies on the other. If any circle through the centre of one cut the other again in P, Q, R, then PQR will form a conjugate triad with respect to the first.

50. A circle through the centre of a rectangular hyperbola cuts the curve in the points A, B, C, D. Prove that the circle circumscribing the triangle formed by the tangents at A, B, C passes through the centre of the hyperbola and has its centre at the point on the hyperbola diametrically opposite to D.

CHAPTER XII.

ENVELOPES.

237. In the general equation of a straight line the two constants are not in any way connected. If however the two constants are connected by any relation, the equation will no longer represent *any* straight line. We have seen, for example, that if the constants l and m in the equation $lx + my - 1 = 0$ satisfy the equation $a^2l^2 + b^2m^2 = 1$, where a and b are known, the line will always touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ [Art. 115]. In every such case, in which the two constants in the equation of a straight line are connected by a relation, the line will touch some curve. This curve is called the *envelope* of the *moving line*.

By means of the relation connecting the two constants we may eliminate one of them, and the equation of the straight line will then contain only one indeterminate constant. If different values be given to this constant we shall have a series of different straight lines all of which will touch some curve.

238. *To find the envelope of a line whose equation contains an indeterminate constant of the second degree.*

Write the equation of the line in the form

$$\mu^2P + 2\mu Q + R = 0 \ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

where μ is the constant.

Through any particular point *two* of the lines will pass, for if the co-ordinates of that point be substituted in (i), we shall have a *quadratic* equation to determine μ. Now the two tangents through any point will be coincident, if the point be on the curve which is touched by the moving line.

Hence, to find the equation of the envelope, we have only to write down the condition that the roots of (i) may be equal, viz. $Q^2 - PR = 0$.

Ex. 1. *To find the envelope of the line*

$$y = mx + \frac{a}{m}.$$

The equation may be written $m^2x - my + a = 0$, and the condition for equal roots gives $y^2 = 4ax$.

Ex. 2. *Find the envelope of a line which cuts off from the axes intercepts whose sum is constant.*

If the equation of the line be $\frac{x}{h} + \frac{y}{k} = 1$, we have $h + k = \text{constant} = c$.

Therefore $\frac{x}{h} + \frac{y}{c - h} = 1$, or $h^2 - h(x - y + c) + xc = 0$. Whence the equation of the envelope is $4cx = (x - y + c)^2$.

Ex. 3. *Find the envelope of the line $ax \cos\theta + by \sin\theta = c$.*

The equation is equivalent to

$$ax - c + 2byt - (ax + c)t^2 = 0, \text{ where } t = \tan\frac{\theta}{2}.$$

Hence, the envelope is

$$(ax - c)(ax + c) + b^2y^2 = 0,$$

or

$$a^2x^2 + b^2y^2 = c^2.$$

Ex. 4. *The envelope of the polar of a given point O with respect to a system of confocal conics is a parabola whose directrix is CO, where C is the centre of the confocals.*

If the confocals be given by the equation

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1,$$

and O be the point (x', y'), the line whose envelope is required is given by

$$\frac{xx'}{a^2 + \lambda} + \frac{yy'}{b^2 + \lambda} = 1,$$

or by

$$\lambda^2 - \lambda(x'x + y'y - a^2 - b^2) + a^2b^2 - b^2x'x - a^2y'y = 0.$$

The equation of the envelope is therefore

$$4\,(a^2b^2 - b^2x'x - a^2y'y) = (x'x + y'y - a^2 - b^2)^2.$$

The envelope is therefore a parabola.

Two confocals pass through O, and the polars of O with respect to them are the tangents at O; hence, since these tangents are at right angles to one another, the point O is on the directrix of the parabola. By considering the limiting forms of the confocals as in Art. 221, we see that the axes themselves are polars of O; hence C is on the directrix of the parabola; so that the directrix is the line CO.

239. *To find the envelope of the line* $lx + my + 1 = 0$, where

$$al^2 + 2hlm + bm^2 + 2gl + 2fm + c = 0.$$

If the line pass through a particular point (x', y') we have $lx' + my' + 1 = 0$. Using this to make the given condition homogeneous in l and m, we have the equation

$$al^2 + 2hlm + bm^2 - 2\,(gl + fm)\,(lx' + my') + c\,(lx' + my')^2 = 0.$$

The two values of the ratio $\dfrac{l}{m}$ give the directions of the two lines which pass through the point (x', y').

If (x', y') be a point on the curve which is touched by the moving line, the tangents from it must be coincident, and therefore the above equation must be a perfect square. The condition for this is

$$(a - 2gx' + cx'^2)\,(b - 2fy' + cy'^2) = (h - gy' - fx' + cx'y')^2,$$

which reduces to

$$x'^2\,(bc - f^2) + 2x'y'\,(fg - ch) + y'^2\,(ca - g^2)$$
$$+ 2x'\,(fh - gb) + 2y'\,(gh - fa) + ab - h^2 = 0.$$

The required envelope is therefore the conic

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$$

where A, B, C, F, G, H mean the same as in Art. 178.

The condition that $lx + my + 1 = 0$ may touch $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$ is $al^2 + 2hlm + bm^2 + 2gl + 2fm + c = 0$. Hence by comparing with the condition found in Art. 178, we see that a, b, c, &c. must be proportional to the minors of A, B, C, &c. in the determinant

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}.$$

This is easily verified, for the minor of A is $BC - F^2$, or

$$(ca - g^2)(ab - h^2) - (gh - af)^2, \text{ that is } a\Delta;$$

and so for the others.

Ex. 1. *To find the envelope of the line* $lx + my + 1 = 0$ *where*

$$al^2 + bm^2 + c = 0.$$

The directions of the lines through (x, y) are given by

$$al^2 + bm^2 + c(lx + my)^2 = 0.$$

These lines will coincide, if

$$(a + cx^2)(b + cy^2) = c^2x^2y^2.$$

Hence the equation of the envelope is

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{1}{c} = 0.$$

Ex. 2. *To find the envelope of the line* $lx + my + 1 = 0$ *with the condition*

$$\frac{f}{l} + \frac{g}{m} + h = 0.$$

The directions of the two lines through (x, y) are given by

$$hlm - (fm + gl)(lx + my) = 0.$$

They will therefore coincide if

$$4fgxy = (fx + gy - h)^2.$$

This is equivalent to

$$\sqrt{fx} + \sqrt{gy} + \sqrt{h} = 0.$$

240. If the equation of a straight line be

$$lx + my + 1 = 0,$$

then the position of the line is determined if l, m are known, and by changing the values of l and m the equation may be made to represent any straight line whatever. The quantities l and m which thus define the position of a line are called the *co-ordinates* of the *line.*

If the co-ordinates of a straight line are connected by any relation the line will envelope a curve, and the equation which expresses the relation is called the *tangential equation* of the curve.

If the tangential equation of the curve is of the nth degree, then n tangents can be drawn to the curve from any point.

Def. A curve is said to be of the nth *class* when n tangents can be drawn to it from a point.

We have seen [Art. 239] that every tangential equation of the second degree represents a conic; also [Art. 178] that the tangential equation of any conic is of the second degree.

If the equation of a straight line be $lx + my + n = 0$, we may call l, m, n the co-ordinates of the line; and if the co-ordinates of the line satisfy any *homogeneous* equation, the line will envelope a curve, of which that equation is called the tangential equation.

241. *To find the director-circle of a conic whose tangential equation is given.*

Let the tangential equation of the conic be

$$al^2 + 2hlm + bm^2 + 2gl + 2fm + c = 0.$$

As in Art. 239, the equation

$$al^2 + 2hlm + bm^2 - 2(gl + fm)(lx + my) + c(lx + my)^2 = 0$$

gives the directions of the two tangents which pass through the particular point (x, y). These tangents will be at right angles to one another if $\frac{l_1}{m_1}\frac{l_2}{m_2} + 1 = 0$, that is, if the sum of the coefficients of l^2 and m^2 is zero.

If therefore (x, y) be a point on the director-circle of the conic, we shall have

$$a - 2gx + cx^2 + b - 2fy + cy^2 = 0 \text{.........(i).}$$

The *centre* of the conic, which coincides with the centre of the director-circle, is the point $\left(\frac{g}{c}, \frac{f}{c}\right)$.

If $c = 0$, the equation (i) is the equation of a straight line. The curve is in this case a parabola, and the equation of its directrix is

$$2gx + 2fy - a - b = 0 \text{............(ii).}$$

In the above we have supposed the axes to be rectangular; if, however, the axes of co-ordinates are inclined to one another at an angle ω, the condition that the straight lines may be at right angles is

$$a - 2gx + cx^2 + b - 2fy + cy^2 + 2\cos\omega\,(h - gy - fx + cxy) = 0.$$

The centre of this circle is $\left(\frac{g}{c}, \frac{f}{c}\right)$.

Hence, whether the axes are rectangular or oblique the centre of the conic, which coincides with the centre of its director-circle, is $\left(\frac{g}{c}, \frac{f}{c}\right)$.

242. *To find the foci of a conic whose tangential equation is given.*

Let (ξ, η) and (ξ', η') be a pair of foci (both being real or both imaginary). The product of the perpendiculars from these points on the line $lx + my + 1 = 0$ will be

$$\frac{(l\xi + m\eta + 1)(l\xi' + m\eta' + 1)}{l^2 + m^2}.$$

This product will be equal to some constant λ for all values of l and m if,

$$l^2(\xi\xi' - \lambda) + lm(\xi\eta' + \eta\xi') + m^2(\eta\eta' - \lambda) + l(\xi + \xi') \\ + m(\eta + \eta') + 1 = 0.$$

Comparing this with the tangential equation we have

$$\frac{\xi\xi' - \lambda}{a} = \frac{\xi\eta' + \eta\xi'}{2h} = \frac{\eta\eta' - \lambda}{b} = \frac{\xi + \xi'}{2g} = \frac{\eta + \eta'}{2f} = \frac{1}{c} \ldots\text{(i)}.$$

Hence $c\xi\xi' - c\eta\eta' = a - b$, and $c\xi\eta' + c\eta\xi' = 2h$.

Eliminate ξ' and η' by means of the last two equations of (i), and we have

$$\xi(c\xi - 2g) - \eta(c\eta - 2f) = b - a,$$

and

$$\xi(c\eta - 2f) + \eta(c\xi - 2g) = -2h.$$

Hence the foci are the four points of intersection of the two conics, $cx^2 - cy^2 - 2gx + 2fy + a - b = 0$,

$$cxy - fx - gy + h = 0.$$

243. If $S = 0$ and $S' = 0$ be the tangential equations of two conics, then $S - \lambda S' = 0$ will be the tangential equation

of a conic touching the four common tangents of the first two.

For, if $S=0$ be

$$al^2+2hlm+bm^2+2gl+2fm+c=0,$$

and $S'=0$ be

$$a'l^2+2h'lm+b'm^2+2g'l+2f'm+c'=0.$$

Then $S-\lambda S'=0$ represents a conic, and any values of l and m which satisfy both $S=0$ and $S'=0$ will, whatever λ may be, satisfy $S-\lambda S'=0$. Therefore the conic $S-\lambda S'=0$ will touch the common tangents of $S=0$ and $S'=0$.

Ex. 1. *The locus of the centres of all conics which touch four fixed straight lines is a straight line.*

If $S=0$, and $S'=0$ be the tangential equations of any two conics which touch the four straight lines, $S-\lambda S'=0$ will be the general equation of the conics touching the lines. The centre of this conic is given by

$$x=\frac{g-\lambda g'}{c-\lambda c'},\quad y=\frac{f-\lambda f'}{c-\lambda c'}. \qquad \text{[Art. 241.]}$$

Eliminating λ we obtain the equation of the centre-locus, viz.

$$x(cf'-c'f)+y(c'g-cg')-f'g+fg'=0.$$

Ex. 2. *The director-circles of all conics which touch four straight lines have a common radical axis.*

The director-circle of the conic $S-\lambda S'=0$ is

$$a+b-2gx-2fy+c(x^2+y^2)-\lambda\{a'+b'-2g'x-2f'y+c'(x^2+y^2)\}=0.$$

[Art. 241.]

This circle always passes through the points common to the two circles

$$x^2+y^2-2\frac{g}{c}x-2\frac{f}{c}y+\frac{a+b}{c}=0,$$

$$x^2+y^2-2\frac{g'}{c'}x-2\frac{f'}{c'}y+\frac{a'+b'}{c'}=0.$$

The radical axis is therefore the line

$$2\left(\frac{g}{c}-\frac{g'}{c'}\right)x+2\left(\frac{f}{c}-\frac{f'}{c'}\right)y-\frac{a+b}{c}+\frac{a'+b'}{c'}=0.$$

One of the conics of the system is a parabola, and its directrix is clearly the common radical axis of the director-circles.

Examples on Chapter XII.

1. PN is the ordinate at any point P of a parabola whose vertex is A, and the rectangle $ANPM$ is completed; find the envelope of the line MN.

2. If the difference of the intercepts on the axes made by a moving line be constant, shew that the line will envelope a parabola.

3. Find the envelope of a straight line which cuts off a constant area from two fixed straight lines.

4. PN, DM are the ordinates of an ellipse at the extremities of a pair of conjugate diameters; find the envelope of PD. Find also the envelope of the line through the middle points of NP and of MD.

5. AB and $A'B'$ are two given finite straight lines, a line PP' cuts these lines so that the ratio $AP : PB$ is equal to $A'P' : P'B'$; shew that PP' envelopes a parabola which touches the given straight lines.

6. OAP, OBQ are two fixed straight lines, A, B are fixed points and P, Q are such that rectangle $AP \,.\, BQ$ is constant, shew that PQ envelopes a conic.

7. A series of circles are described each touching two given straight lines; shew that the polars of any given point with respect to the circles will envelop a parabola.

8. Two points are taken on an ellipse such that the sum of the ordinates is constant; shew that the envelope of the line joining the points is a parabola.

9. A fixed tangent to a parabola is cut by any other tangent PT in the point T, and TQ is drawn perpendicular to TP; shew that TQ envelopes another parabola.

10. Through any point P on a given straight line a line PQ is drawn parallel to the polar of P with respect to a given conic; prove that the envelope of these lines is a parabola.

11. If a leaf of a book be folded so that one corner moves along an opposite side, the line of the crease will envelope a parabola.

12. An ellipse turns about its centre, find the envelope of the chords of intersection with the initial position.

13. An angle of constant magnitude moves so that one side passes through a fixed point and its summit moves along a fixed straight line; shew that the other side envelopes a parabola.

14. The middle point of a chord PQ of an ellipse is on a given straight line; shew that the chord PQ envelopes a parabola.

15. O is any point on a conic and OP, OQ are chords drawn parallel to two fixed straight lines; shew that PQ envelopes a conic.

16. Any pair of conjugate diameters of an ellipse meets a fixed circle concentric with the ellipse in P, Q; shew that PQ will envelope a similar and similarly situated ellipse.

17. If the sum of the squares of the perpendiculars from any number of fixed points on a straight line be constant; shew that the line will envelope a conic.

18. The sides of a triangle, produced if necessary, are cut by a straight line in the points L, M, N respectively; shew that, if $LM : MN$ be constant the line will envelope a parabola.

19. OA, OB are two fixed straight lines, and a circle which passes through O and through another given fixed point cuts the lines in P, Q respectively; shew that the line PQ envelopes a parabola.

20. The four normals to an ellipse at P, Q, R, S meet in a point; prove that if the chord PQ pass through a fixed point, the chord RS will envelope a parabola.

21. A rectangular hyperbola is cut by a circle of any radius whose centre is at a fixed point on one of the axes of the hyperbola; shew that the lines joining the points of intersection are either parallel to an axis of the hyperbola or are tangents to a fixed parabola.

22. Shew that the envelope of the polar of a given point with respect to a system of ellipses whose axes are given in magnitude and direction and whose centres are on a given straight line is a parabola.

23. Of two equal circles one is fixed and the other passes through a fixed point; shew that their radical axis envelopes a conic having the fixed point for focus.

24. If pairs of radii vectors be drawn from the centre of an ellipse making with the major axis angles whose sum is a right angle, the locus of the poles of the chords joining their extremities is a concentric hyperbola, and the envelope of the chords is a rectangular hyperbola.

25. From any point on one of the equi-conjugate diameters of a conic lines are drawn to the extremities of an axis and these lines cut the curve again in the points P, Q; shew that the envelope of PQ is a rectangular hyperbola.

26. PNP' is the double ordinate of an ellipse which is equi-distant from the centre C and a vertex; shew that if parabolas be drawn through P, P', C, the chords joining the other intersections of the parabola and ellipse will touch a second ellipse equal in all respects to the given one.

27. Two given parallel straight lines are cut in the points P, Q by a line which passes through a fixed point; find the envelope of the circle on PQ as diameter.

28. The envelope of the circles described on a system of parallel chords of a conic as diameters is another conic.

29. A chord of a parabola is such that the circle described on the chord as diameter will touch the curve; shew that the chord envelopes another parabola.

30. Shew that the envelope of the directrices of all parabolas which have a common vertex A, and which pass through a fixed point P, is a parabola the length of whose latus rectum is AP.

31. Prove that, if the bisectors of the internal and external angles between two tangents to a conic be parallel to two given diameters of the conic, the chord of contact will envelope an hyperbola whose asymptotes are the conjugates of those diameters.

32. The polar of a point P with respect to a given conic S meets two fixed straight lines AB, AC in Q, Q'; shew that, if AP bisect QQ', the locus of P will be a conic; shew also that the envelope of QQ' will be another conic.

33. If two points be taken on a conic so that the harmonic mean of their distances from one focus is constant, shew that the chord joining them will always touch a confocal conic.

34. The envelope of the chord of a parabola which subtends a right angle at the focus is the ellipse

$$(x-3a)^2+2y^2=8a^2,\ y^2-4ax=0$$

being the equation of the parabola.

35. A chord of a conic which subtends a constant angle at a given point on the curve envelopes a conic having double contact with the given conic.

36. Through a fixed point a pair of chords of a circle are drawn at right angles; prove that each side of the quadrilateral formed by joining their extremities envelope a conic of which the fixed point and the centre of the circle are foci.

37. The perpendicular from a point S on its polar with respect to a parabola meets the axis of the parabola in C; shew that chords of the parabola which subtend a right angle at S all touch a conic whose centre is C.

38. Shew that chords of a conic which subtend a right angle at a fixed point O envelope another conic.

Shew also that the point O is a focus of the envelope and that the directrix corresponding to O is the polar of O with respect to the original conic.

Shew that the envelopes corresponding to a system of concentric similar and similarly situated conics are confocal.

39. A straight line meets one of a system of confocal conics in P, Q, and RS is the line joining the feet of the other two normals drawn from the point of intersection of the normals at P and Q. Prove that the envelope of RS is a parabola touching the axes.

40. If a line cut two given circles so that the portions of the line intercepted by the circles are in a constant ratio, shew that it will envelope a conic, which will be a parabola if the ratio be one of equality.

41. Chords of a rectangular hyperbola at right angles to each other, subtend right angles at a fixed point O; shew that they intersect in the polar of O.

42. Shew that if AP, AQ be two chords of the parabola $y^2 - 4ax = 0$ through the vertex A, which make an angle $\frac{\pi}{4}$ with one another; the line PQ will always touch the ellipse

$$\frac{(x-12a)^2}{128a^2} + \frac{y^2}{16a^2} = 1.$$

43. Pairs of points are taken on a conic, such that the lines joining them to a given point are equally inclined to a given straight line; prove that the chord joining any such pair of points envelopes a conic whose director-circle passes through the fixed point.

44. Chords of a conic S which subtend a right angle at a fixed point envelope a conic S'. Shew that, if S pass through four fixed points, S' will touch four fixed straight lines.

45. A conic passes through the four fixed points A, B, C, D and the tangents to it at B and C are met by CA, BA produced in P, Q. Shew that PQ envelopes a conic which touches BA, CA.

46. If a chord cut a circle in two points A, B which are such that the rectangle $OA \,.\, OB$ is constant, O being a fixed point; shew that the envelope of the chord is a conic of which O is a focus. Shew also that if $OA^2 + OB^2$ be constant, the chord will envelope a parabola.

47. On a diameter of a circle two points A, A' are taken equally distant from the centre, and the lines joining any point P of the circle to these points cut the circle again in Q, R; shew that QR envelopes a conic of which the given circle is the auxiliary circle.

48. A triangle is inscribed in an ellipse and two of its sides pass through fixed points; shew that the envelope of the third side is a conic having double contact with the former.

49. A triangle is inscribed in an ellipse and two of its sides touch a coaxial ellipse; shew that the envelope of the third side is a third ellipse.

50. Shew that the locus of the centre of a conic which is inscribed in a given triangle, and which has the sum of the squares of its axes constant, is a circle.

CHAPTER XIII.

TRILINEAR CO-ORDINATES.

244. LET any three straight lines be taken which do not meet in a point, and let ABC be the triangle formed by them. Let the perpendicular distances of any point P from the sides BC, CA, AB be α, β, γ respectively; then α, β, γ are called the trilinear co-ordinates of the point P referred to the triangle ABC. We shall consider α, β, γ to be positive when drawn in the same direction as the perpendiculars on the sides from the opposite angular points of the triangle of reference.

Two of these perpendicular distances are sufficient to determine the position of any point, there must therefore be some relation connecting the three.

The relation is

$$a\alpha + b\beta + c\gamma = 2\Delta,$$

where Δ is the area of the triangle ABC. This is evidently true for any point P *within* the triangle, since the triangles BPC, CPA and APB are together equal to the triangle ABC; and, regard being had to the signs of the perpendiculars, it can be easily seen to be universally true, by drawing figures for the different cases.

245. By means of the relation $a\alpha + b\beta + c\gamma = 2\Delta$ any equation can be made homogeneous in α, β, γ; and when we have done this we may use instead of the actual co-ordinates of a point, any quantities proportional to them: for if any values α, β, γ satisfy a homogeneous equation, then $k\alpha$, $k\beta$, $k\gamma$ will also satisfy that equation.

246. If any origin be taken within the triangle, the equations of the sides of the triangle referred to any rectangular axes through this point can be written in the form

$$-x\cos\theta_1 - y\sin\theta_1 + p_1 = 0,$$
$$-x\cos\theta_2 - y\sin\theta_2 + p_2 = 0,$$
$$-x\cos\theta_3 - y\sin\theta_3 + p_3 = 0,$$

where $\cos(\theta_2 - \theta_3) = -\cos A$, $\cos(\theta_3 - \theta_1) = -\cos B$,
and $$\cos(\theta_1 - \theta_2) = -\cos C.$$

[We write the equations with the constant terms positive because the perpendiculars on the sides from a point within the triangle are all positive.]

We therefore have [Art. 31]

$$\alpha = p_1 - x\cos\theta_1 - y\sin\theta_1,$$
$$\beta = p_2 - x\cos\theta_2 - y\sin\theta_2,$$
and $$\gamma = p_3 - x\cos\theta_3 - y\sin\theta_3.$$

By means of the above we can change any equation in trilinear co-ordinates into the corresponding equation in common (or Cartesian) co-ordinates.

247. *Every equation of the first degree represents a straight line.*

Let the equation be

$$l\alpha + m\beta + n\gamma = 0.$$

If we substitute the values found in the preceding Article for α, β, γ, the equation in Cartesian co-ordinates so found will clearly be of the first degree. Therefore the locus is a straight line.

248. *Every straight line can be represented by an equation of the first degree.*

It will be sufficient to shew that we can always find values of l, m, n such that the equation $l\alpha + m\beta + n\gamma = 0$, which we know represents a straight line, is satisfied by the co-ordinates of any two points.

If the co-ordinates of the points be α', β', γ' and

α'', β'', γ'' we must have

$$l\alpha' + m\beta' + n\gamma' = 0,$$
$$l\alpha'' + m\beta'' + n\gamma'' = 0,$$

and values of l, m, n can always be found to satisfy these two equations.

249. *To find the equation of a straight line which passes through two given points.*

Let α', β', γ'; α'', β'', γ'' be the co-ordinates of the two points.

The equation of any straight line is

$$l\alpha + m\beta + n\gamma = 0.$$

The points $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$, are on the line if

$$l\alpha' + m\beta' + n\gamma' = 0,$$
$$l\alpha'' + m\beta'' + n\gamma'' = 0.$$

Eliminating l, m, n from these three equations we have

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} = 0.$$

250. *To find the condition that three given points may be on a straight line.*

Let the co-ordinates of the given points be α', β', γ'; α'', β'', γ''; and α''', β''', γ'''.

If these are on the straight line whose equation is

$$l\alpha + m\beta + n\gamma = 0,$$

we must have
$$l\alpha' + m\beta' + n\gamma' = 0,$$
$$l\alpha'' + m\beta'' + n\gamma'' = 0,$$
and
$$l\alpha''' + m\beta''' + n\gamma''' = 0.$$

Eliminating l, m, n we obtain the required condition, viz.

$$\begin{vmatrix} \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \\ \alpha''', & \beta''', & \gamma''' \end{vmatrix} = 0.$$

251. *To find the point of intersection of two given straight lines.*

Let the equations of the given straight lines be

$$l\alpha + m\beta + n\gamma = 0,$$

and

$$l'\alpha + m'\beta + n'\gamma = 0.$$

At the point which is common to these, we have

$$\frac{\alpha}{mn' - m'n} = \frac{\beta}{nl' - n'l} = \frac{\gamma}{lm' - l'm} \ldots\ldots\ldots\ldots (\text{i}).$$

The above equations give the *ratios* of the co-ordinates.

If the actual values be required, multiply the numerators and denominators of the fractions in (i) by a, b, c respectively, and add; then each fraction is equal to

$$\frac{a\alpha + b\beta + c\gamma}{a(mn' - m'n) + b(nl' - n'l) + c(lm' - l'm)}, \text{ or } \frac{2\Delta}{\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ a, & b, & c \end{vmatrix}}.$$

The lines will not meet in a point at a finite distance from the triangle of reference, that is to say the lines will be parallel, if

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ a, & b, & c \end{vmatrix} = 0.$$

252. *To find the condition that three straight lines may meet in a point.*

Let the equations of the straight lines be

$$l_1\alpha + m_1\beta + n_1\gamma = 0,$$
$$l_2\alpha + m_2\beta + n_2\gamma = 0,$$
$$l_3\alpha + m_3\beta + n_3\gamma = 0.$$

The lines will meet in a point if the above equations are all satisfied by the same values of α, β, γ. The elimination of α, β, γ gives for the required condition

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} = 0.$$

253. If $Ax+By+C=0$ be the equation of a straight line in Cartesian co-ordinates, the intercepts which the line makes on the axes are $-\frac{C}{A}$, $-\frac{C}{B}$ respectively. If therefore A and B be *very small* the line will be at a *very great* distance from the origin. The equation of the line will in the limit, assume the form

$$0 \,.\, x+0 \,.\, y+C=0.$$

The equation of an infinitely distant straight line, generally called the *line at infinity*, is therefore

$$0 \,.\, x+0 \,.\, y+C=0.$$

When the line at infinity is to be combined with other expressions involving x and y it is written $C=0$.

The equation of the line at infinity in trilinear co-ordinates is $$a\alpha+b\beta+c\gamma=0.$$

For if $k\alpha$, $k\beta$, $k\gamma$ be the co-ordinates of any point, the invariable relation gives $k(a\alpha+b\beta+c\gamma)=2\Delta$, or

$$a\alpha+b\beta+c\gamma=\frac{2\Delta}{k}.$$

If therefore k become infinitely great, we have in the limit the relation $a\alpha+b\beta+c\gamma=0$. This is a linear relation which is satisfied by finite quantities which are proportional to the co-ordinates of *any* infinitely distant point, and it is not satisfied by the co-ordinates, or by quantities proportional to the co-ordinates, of any point at a finite distance from the triangle of reference.

254. *To find the condition that two given lines may be parallel.*

Let the equations of the lines be

$$l\alpha+m\beta+n\gamma=0,$$
$$l'\alpha+m'\beta+n'\gamma=0.$$

If the lines are parallel their point of intersection will be at an infinite distance from the origin and therefore its co-ordinates will satisfy the relation

$$a\alpha+b\beta+c\gamma=0.$$

Eliminating α, β, γ from the three equations, we have the required condition, viz.

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ a, & b, & c \end{vmatrix} = 0.$$

255. *To find the equation of a straight line through a given point parallel to a given straight line.*

Let the equation of the given line be

$$l\alpha + m\beta + n\gamma = 0.$$

The required line meets this where

$$a\alpha + b\beta + c\gamma = 0.$$

The equation is therefore of the form

$$l\alpha + m\beta + n\gamma + \lambda (a\alpha + b\beta + c\gamma) = 0.$$

If f, g, h be the co-ordinates of the given point, we must also have

$$lf + mg + nh + \lambda (af + bg + ch) = 0,$$

whence $$\frac{l\alpha + m\beta + n\gamma}{lf + mg + nh} = \frac{a\alpha + b\beta + c\gamma}{af + bg + ch}.$$

A useful case is to find the equation of a straight line through an angular point of the triangle of reference parallel to a given straight line.

If A be the angular point, its co-ordinates are $f, 0, 0$, and the equation becomes $(ma - lb)\,\beta + (na - lc)\,\gamma = 0$.

256. *To find the condition of perpendicularity of two given straight lines.*

Let the equations of the lines be

$$l\alpha + m\beta + n\gamma = 0,$$
$$l'\alpha + m'\beta + n'\gamma = 0.$$

If these be expressed in Cartesian co-ordinates by means of the equations found in Art. 246, they will be

$$x(l\cos\theta_1 + m\cos\theta_2 + n\cos\theta_3) + y(l\sin\theta_1 + m\sin\theta_2 + n\sin\theta_3) - lp_1 - mp_2 - np_3 = 0,$$

and

$$x(l'\cos\theta_1+m'\cos\theta_2+n'\cos\theta_3)+y(l'\sin\theta_1+m'\sin\theta_2+n'\sin\theta_3) - l'p_1-m'p_2-n'p_3=0:$$

the lines will therefore be perpendicular [Art. 29] if

$$(l\cos\theta_1+m\cos\theta_2+n\cos\theta_3)(l'\cos\theta_1+m'\cos\theta_2+n'\cos\theta_3) + (l\sin\theta_1+m\sin\theta_2+n\sin\theta_3)(l'\sin\theta_1+m'\sin\theta_2+n'\sin\theta_3)=0;$$

that is, if

$$ll'+mm'+nn'+(lm'+l'm)\cos(\theta_1\sim\theta_2) + (mn'+m'n)\cos(\theta_2\sim\theta_3)+(nl'+n'l)\cos(\theta_3\sim\theta_1)=0.$$

But $\cos(\theta_2-\theta_3)=-\cos A$, $\cos(\theta_3-\theta_1)=-\cos B$,

and $\cos(\theta_1-\theta_2)=-\cos C$;

therefore the required condition is

$$ll'+mm'+nn'-(mn'+m'n)\cos A-(nl'+n'l)\cos B - (lm'+l'm)\cos C=0.$$

257. *To find the perpendicular distance of a given point from a given straight line.*

Let the equation of the straight line be

$$l\alpha+m\beta+n\gamma=0.$$

Expressed in Cartesian co-ordinates the equation will be

$$x(l\cos\theta_1+m\cos\theta_2+n\cos\theta_3)+y(l\sin\theta_1+m\sin\theta_2+n\sin\theta_3) - lp_1-mp_2-np_3=0.$$

The perpendicular distance of any point from this line is found by substituting the co-ordinates of the point in the expression on the left of the equation and dividing by the square root of the sum of the squares of the coefficients of x and y. If this be again expressed in trilinear co-ordinates, we shall have, for the length of the perpendicular from f, g, h on the given line, the value

$$\frac{lf+mg+nh}{\sqrt{\{(l\cos\theta_1+m\cos\theta_2+n\cos\theta_3)^2+(l\sin\theta_1+m\sin\theta_2+n\sin\theta_3)^2\}}}.$$

The denominator is the square root of

$$l^2+m^2+n^2+2lm\cos(\theta_1-\theta_2)+2mn\cos(\theta_2-\theta_3) + 2nl\cos(\theta_3-\theta_1),$$

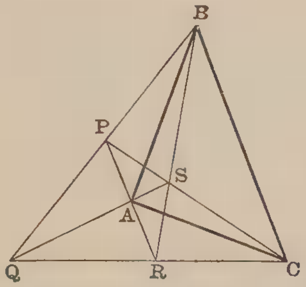
or of $\quad l^2+m^2+n^2-2lm\cos C-2mn\cos A-2nl\cos B.$

Hence the length of the perpendicular is equal to

$$\frac{lf+mg+nh}{\sqrt{(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)}}.$$

258. *To shew that the co-ordinates of any four points may be expressed in the form* $\pm f$, $\pm g$, $\pm h$.

Let P, Q, R, S be the four points.



The intersection of the line joining two of the points and the line joining the other two is called a *diagonal-point* of the quadrangle. There are therefore three diagonal-points, viz. the points A, B, C in the figure.

Take ABC for the triangle of reference, and let the co-ordinates of P be f, g, h.

Then the equation of AP will be $\frac{\beta}{g}=\frac{\gamma}{h}$.

The pencil AB, AS, AC, AP is harmonic [Art. 60], and the equations of AB, AC are $\gamma=0$, $\beta=0$ respectively, and the equation of AP is $\frac{\beta}{g}=\frac{\gamma}{h}$; therefore the equation of AS will be $\frac{\beta}{g}=\frac{\gamma}{-h}$. [Art. 56.]

The equation of CP is $\frac{\alpha}{f}=\frac{\beta}{g}$.

Therefore where AS and CP meet, i.e. at S, we shall have

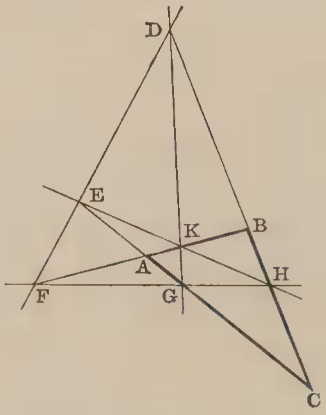
$$\frac{\alpha}{f}=\frac{\beta}{g}=\frac{\gamma}{-h}.$$

So that the co-ordinates of S are proportional to $f, g, -h$.
Similarly the co-ordinates of R are proportional to $-f, g, h$.
Similarly the co-ordinates of Q are proportional to $f, -g, h$.

259. *To shew that the equations of any four straight lines may be expressed in the form* $l\alpha \pm m\beta \pm n\gamma = 0$.

Let DEF, DKG, EKH, FGH be the four straight lines.

Let ABC be the triangle formed by the diagonals FK, EG, and DH of the quadrilateral, and take ABC for the triangle of reference.



Let the equation of DEF be

$$l\alpha + m\beta + n\gamma = 0.$$

Then the equation of AD is $m\beta + n\gamma = 0$.

Since the pencil AD, AB, AH, AC is harmonic [Art. 60], and the equations of AD, AB, AC are $m\beta + n\gamma = 0$, $\gamma = 0$, $\beta = 0$ respectively;
therefore [Art. 56] the equation of AH is $m\beta - n\gamma = 0$.

Since E is the point given by $\beta = 0$, $l\alpha + n\gamma = 0$; and H is the point given by $\alpha = 0$, $m\beta - n\gamma = 0$; the equation of HE is $$l\alpha - m\beta + n\gamma = 0.$$

We can shew in a similar manner that the equation of DK is
$$-l\alpha + m\beta + n\gamma = 0,$$
and that the equation of FH is
$$l\alpha + m\beta - n\gamma = 0.$$

EXAMPLES.

1. The three bisectors of the angles of the triangle of reference have for equations, $\beta - \gamma = 0$, $\gamma - \alpha = 0$, and $\alpha - \beta = 0$.

2. The three straight lines from the angular points of the triangle of reference to the middle points of the opposite sides have for equations $b\beta - c\gamma = 0$, $c\gamma - a\alpha = 0$, and $a\alpha - b\beta = 0$.

3. If $A'B'C'$ be the middle points of the sides of the triangle of reference, the equations of $B'C'$, $C'A'$, $A'B'$ will be $b\beta + c\gamma - a\alpha = 0$, $c\gamma + a\alpha - b\beta = 0$, $a\alpha + b\beta - c\gamma = 0$ respectively.

4. The equation of the line joining the centres of the inscribed and circumscribed circles of a triangle is
$$\alpha(\cos B - \cos C) + \beta(\cos C - \cos A) + \gamma(\cos A - \cos B) = 0.$$

5. Find the co-ordinates of the centres of the four circles which touch the sides of the triangle of reference. Find also the co-ordinates of the six middle points of the lines joining the four centres, and shew that the co-ordinates of these six points all satisfy the equation
$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0.$$

6. If AO, BO, CO meet the sides of the triangle ABC in A', B', C'; and if $B'C'$ meet BC in P, $C'A'$ meet CA in Q, and $A'B'$ meet AB in R; shew that P, Q, R are on a straight line.

Shew also that BQ, CR, AA' meet in a point P'; CR, AP, BB' meet in a point Q'; and that AP, BQ, CC' meet in a point R'.

7. If through the middle points, A', B', C' of the sides of the triangle ABC lines $A'P$, $B'Q$, $C'R$ be drawn perpendicular to the sides and equal to them; shew that AP, BQ, CR will meet in a point.

8. If p, q, r be the lengths of the perpendiculars from the angular points of the triangle of reference on any straight line; shew that the equation of the line will be $ap\alpha + bq\beta + cr\gamma = 0$.

9. If there be two triangles such that the straight lines joining the corresponding angles meet in a point, then will the three intersections of corresponding sides lie on a straight line.

[Let f, g, h be the co-ordinates of the point, referred to ABC one of the two triangles. Then the co-ordinates of the angular points of the other triangle $A'BC'$ can be taken to be f', g, h; f, g', h and f, g, h' respectively. $B'C'$ cuts BC where $\alpha = 0$ and $\dfrac{\beta}{g - g'} + \dfrac{\gamma}{h - h'} = 0$. Hence the three intersections of corresponding sides are on the line $\dfrac{\alpha}{f - f'} + \dfrac{\beta}{g - g'} + \dfrac{\gamma}{h - h'} = 0$.]

260. The general equation of the second degree in trilinear co-ordinates, viz.

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0,$$

is the equation of a conic section; for, if the equation be expressed in Cartesian co-ordinates the equation will be of the second degree.

Also, since the equation contains five independent constants, these can be so determined that the curve represented by the equation will pass through five given points, and therefore will coincide with any given conic.

261. *To find the equation of the tangent at any point of a conic.*

Let the equation of the conic be

$$\phi(\alpha, \beta, \gamma) \equiv u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0,$$

and let α', β', γ'; $\alpha'', \beta'', \gamma''$ be the co-ordinates of two points on it.

The equation

$$\begin{aligned} u(\alpha-\alpha')(\alpha-\alpha'') + v(\beta-\beta')(\beta-\beta'') + w(\gamma-\gamma')(\gamma-\gamma'') \\ + 2u'(\beta-\beta')(\gamma-\gamma'') + 2v'(\gamma-\gamma')(\alpha-\alpha'') \\ + 2w'(\alpha-\alpha')(\beta-\beta'') = \phi(\alpha, \beta, \gamma), \end{aligned}$$

is really of the first degree in α, β, γ, and therefore it is the equation of *some* straight line. The equation is satisfied by the values $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, and also by the values $\alpha = \alpha''$, $\beta = \beta''$, $\gamma = \gamma''$. Therefore it is the equation of the line joining the two points $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$. Let now $(\alpha'', \beta'', \gamma'')$ move up to and ultimately coincide with $(\alpha', \beta', \gamma')$, and we have the equation of the *tangent* at $(\alpha', \beta', \gamma')$, viz.,

$$\begin{aligned} u\alpha\alpha' + v\beta\beta' + w\gamma\gamma' + u'(\beta\gamma' + \gamma\beta') \\ + v'(\gamma\alpha' + \alpha\gamma') + w'(\alpha\beta' + \beta\alpha') = 0. \end{aligned}$$

Using the notation of the Differential Calculus we may write the equation of the tangent at any point $(\alpha', \beta', \gamma')$

of the conic $\phi(\alpha, \beta, \gamma) = 0$ in either of the forms

$$\alpha \frac{d\phi}{d\alpha'} + \beta \frac{d\phi}{d\beta'} + \gamma \frac{d\phi}{d\gamma'} = 0,$$

or

$$\alpha' \frac{d\phi}{d\alpha} + \beta' \frac{d\phi}{d\beta} + \gamma' \frac{d\phi}{d\gamma} = 0.$$

262. *To find the equation of the polar of a given point.*

It may be shewn, exactly as in Art. 76, 100, or 118, that the equation of the polar of a point with respect to a conic is of the *same form* as the equation we have found for the tangent in Art. 261.

263. *To find the condition that a given straight line may touch a conic.*

Let the equation of the given straight line be

$$l\alpha + m\beta + n\gamma = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

The equation of the tangent at $(\alpha', \beta', \gamma')$ is

$$\alpha(u\alpha' + w'\beta' + v'\gamma') + \beta(w'\alpha' + v\beta' + u'\gamma') \\ + \gamma(v'\alpha' + u'\beta' + w\gamma') = 0 \ldots\text{(ii)}.$$

If (i) and (ii) represent the same straight line, we have

$$\frac{u\alpha' + w'\beta' + v'\gamma'}{l} = \frac{w'\alpha' + v\beta' + u'\gamma'}{m} = \frac{v'\alpha' + u'\beta' + w\gamma'}{n}.$$

Putting each of these fractions equal to $-\lambda$, we have

$$\begin{aligned} u\alpha' + w'\beta' + v'\gamma' + \lambda l &= 0, \\ w'\alpha' + v\beta' + u'\gamma' + \lambda m &= 0, \\ v'\alpha' + u'\beta' + w\gamma' + \lambda n &= 0. \end{aligned}$$

Also, since $(\alpha', \beta', \gamma')$ is on the line (l, m, n),

$$l\alpha' + m\beta' + n\gamma' = 0.$$

Eliminating $\alpha', \beta', \gamma', \lambda$ from these four equations we obtain the required condition

$$\begin{vmatrix} u, & w', & v', & l \\ w', & v, & u', & m \\ v', & u', & w, & n \\ l, & m, & n, & 0 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\text{(iii)},$$

or $$l^2 . (vw - u'^2) + m^2 (wu - v'^2) + n^2 (uv - w'^2)$$
$$+ 2mn (v'w' - uu') + 2nl (w'u' - vv') + 2lm (u'v' - ww') = 0,$$
or
$$Ul^2 + Vm^2 + Wn^2 + 2U'mn + 2V'nl + 2W'lm = 0 \ldots (\text{v}),$$
where U, V, W, U', V', W' are the minors of u, v, w, u', v', w' in the determinant
$$\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} = 0.$$

264. *To find the co-ordinates of the centre of a conic.*

Since the polar of the centre of a conic is altogether at an infinite distance, its equation is
$$a\alpha + b\beta + c\gamma = 0 \ldots\ldots\ldots\ldots (\text{i}).$$

But [Art. 262], the equation of the polar of the centre will be
$$\alpha \frac{d\phi}{d\alpha_0} + \beta \frac{d\phi}{d\beta_0} + \gamma \frac{d\phi}{d\gamma_0} = 0,$$
where α_0, β_0, γ_0, are the co-ordinates of the centre.

Hence the equations for finding the centre are
$$\frac{\frac{d\phi}{d\alpha_0}}{a} = \frac{\frac{d\phi}{d\beta_0}}{b} = \frac{\frac{d\phi}{d\gamma_0}}{c}.$$

265. *To find the condition that the curve represented by the general equation of the second degree may be a parabola.*

The co-ordinates of the centre of the curve are given by the equations
$$\frac{u\alpha_0 + w'\beta_0 + v'\gamma_0}{a} = \frac{w'\alpha_0 + v\beta_0 + u'\gamma_0}{b} = \frac{v'\alpha_0 + u'\beta_0 + w\gamma_0}{c}.$$

Put each of these equal to $-\lambda$, and we have
$$u\alpha_0 + w'\beta_0 + v'\gamma_0 + \lambda a = 0,$$
$$w'\alpha_0 + v\beta_0 + u'\gamma_0 + \lambda b = 0,$$
$$v'\alpha_0 + u'\beta_0 + w\gamma_0 + \lambda c = 0.$$

Also since the centre of a parabola is at infinity, we have

$$a\alpha_0 + b\beta_0 + c\gamma_0 = 0.$$

The elimination of α_0, β_0, γ_0, λ gives for the required condition

$$\begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c, & 0 \end{vmatrix} = 0.$$

We see from the above that the parabola touches the line at infinity. [Art. 263.]

266. *To find the condition that the conic represented by the general equation of the second degree may be two straight lines.*

The required condition may be found as in Art. 37. The condition is

$$uvw + 2u'v'w' - uu'^2 - vv'^2 - ww'^2 = 0,$$

or, as a determinant,

$$\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} = 0.$$

267. *To find the asymptotes of a conic.*

The equations of the curve and of its asymptotes only differ by a constant.

Hence if the equation of the curve be

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0,$$

the equation of the asymptotes will be

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta \\ + \lambda(a\alpha + b\beta + c\gamma)^2 = 0 \text{........(i)}.$$

The value of λ is to be determined from the condition for straight lines, viz.

$$\begin{vmatrix} u + \lambda a^2, & w' + \lambda ab, & v' + \lambda ac \\ w' + \lambda ab, & v + \lambda b^2, & u' + \lambda bc \\ v' + \lambda ac, & u' + \lambda bc, & w + \lambda c^2 \end{vmatrix} = 0.$$

The term independent of λ is

$$\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}.$$

The coefficient of λ is

$$\begin{vmatrix} a^2, & ab, & ac \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} + \begin{vmatrix} u, & w', & v' \\ ab, & b^2, & bc \\ v', & u', & w \end{vmatrix} + \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ ac, & bc, & c^2 \end{vmatrix},$$

which is equal to

$$- \begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c, & 0 \end{vmatrix}.$$

The coefficients of λ^2 and of λ^3 are both zero.

Hence there is a simple equation for λ, and therefore from (i) we have for the equation of the asymptotes

$$\phi(\alpha, \beta, \gamma) \begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c, & 0 \end{vmatrix} + (a\alpha + b\beta + c\gamma)^2 \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} = 0.$$

268. *To find the condition that the conic may be a rectangular hyperbola.*

Change to Cartesian co-ordinates. Then the conic will be a rectangular hyperbola if the sum of the coefficients of x^2 and y^2 is zero.

The condition becomes

$$u + v + w - 2u' \cos A - 2v' \cos B - 2w' \cos C = 0.$$

269. *To find the equation of the circle circumscribing the triangle of reference.*

If from any point P, on the circle circumscribing a triangle ABC, the three perpendiculars PL, PM, PN be drawn to the sides of the triangle and meet the sides

BC, *CA*, *AB* in the points *L*, *M*, *N* respectively; then it is known that these three points *L*, *M*, *N* are in a straight line.

Let the triangle be taken for the triangle of reference and let α, β, γ be the co-ordinates of *P*.

The areas of the triangles *MPN*, *NPL*, and *LPM* are $\frac{1}{2}\beta\gamma \sin A$, $\frac{1}{2}\gamma\alpha \sin B$, and $\frac{1}{2}\alpha\beta \sin C$ respectively. Since *L*, *M*, *N* are on a straight line, one of these triangles is equal to the sum of the other two. Hence, regard being had to sign, we have

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0,$$

or

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0,$$

which is the equation required.

Ex. The perpendiculars from *O* on the sides of a triangle meet the sides in *D*, *E*, *F*. Shew that, if the area of the triangle *DEF* is constant, the locus of *O* is a circle concentric with the circumscribing circle.

270. Since the terms of the second degree are the same in the equations of all circles, if $S = 0$ be the equation of any one circle, the equation of any other circle can be written in the form

$$S + \lambda\alpha + \mu\beta + \nu\gamma = 0,$$

or, in the homogeneous form,

$$S + (l\alpha + m\beta + n\gamma)(a\alpha + b\beta + c\gamma) = 0.$$

271. From the form of the general equation of a circle in Art. 270 it is evident that the line at infinity cuts all circles in the same two (imaginary) points.

The two points at infinity through which all circles pass are called the *circular points at infinity*.

Since, in Cartesian co-ordinates, the lines $x^2 + y^2 = 0$ are parallel to the asymptotes of any circle, the imaginary lines $x^2 + y^2 = 0$ go through the circular points at infinity. Hence, from Art. 193, the four points of intersection of the tangents drawn to a conic from the circular points at infinity are the four foci of the curve.

272. *To find the conditions that the curve represented by the general equation of the second degree may be a circle.*

The equation of the circle circumscribing the triangle of reference is [Art. 269]

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0.$$

Therefore [Art. 270] the equation of any other circle is of the form

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + (l\alpha + m\beta + n\gamma)(a\alpha + b\beta + c\gamma) = 0.$$

If this is the same curve as that represented by

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0,$$

we must have, for some value of λ,

$$\lambda u = la,\ \lambda v = mb,\ \lambda w = nc,$$

$$2\lambda u' = a + cm + bn, 2\lambda v' = b + an + cl, \text{and } 2\lambda w' = c + bl + am.$$

Hence

$$2bcu' - c^2v - b^2w = 2cav' - a^2w - c^2u = 2abw' - b^2u - a^2v,$$

for each of these quantities is equal to $\dfrac{abc}{\lambda}$.

273. *To find the condition that the conic represented by the general equation of the second degree may be an ellipse, parabola, or hyperbola.*

The equation of the lines from the angular point C to the points at infinity on the conic will be found by eliminating γ from the equation of the curve and the equation $a\alpha + b\beta + c\gamma = 0$. Hence the equation of the lines through C parallel to the asymptotes of the conic will be

$$uc^2\alpha^2 + vc^2\beta^2 + w(a\alpha + b\beta)^2 - 2u'c\beta(a\alpha + b\beta)$$
$$- 2v'c\alpha(a\alpha + b\beta) + 2w'c^2\alpha\beta = 0.$$

The conic is an ellipse, parabola, or hyperbola, according as these lines are imaginary, coincident, or real; and the lines are imaginary, coincident, or real according as

$$(wab - u'ac - v'bc + w'c^2)^2 - (uc^2 + wa^2 - 2v'ac)$$
$$(vc^2 + wb^2 - 2u'bc)$$

is negative, zero, or positive; that is, according as

$$Ua^2 + Vb^2 + Wc^2 + 2U'bc + 2V'ca + 2W'ab$$

is positive, zero, or negative.

274. The equation of a *pair of tangents* drawn to the conic from any point can be found by the method of Art. 188.

The equation of the *director circle* of the conic can be found by the method of Art. 189.

The equation giving the *foci* can be found by the method of Art. 193.

The equations for the foci will be found to be

$$4(b^2w + c^2v - 2bcu')\,\phi(\alpha, \beta, \gamma) - \left(b\frac{d\phi}{d\gamma} - c\frac{d\phi}{d\beta}\right)^2$$
$$= 4(c^2u + a^2w - 2cav')\,\phi(\alpha, \beta, \gamma) - \left(c\frac{d\phi}{d\alpha} - a\frac{d\phi}{d\gamma}\right)^2$$
$$= 4(a^2v + b^2u - 2abw')\,\phi(\alpha, \beta, \gamma) - \left(a\frac{d\phi}{d\beta} - b\frac{d\phi}{d\alpha}\right)^2.$$

The elimination of $\phi(\alpha, \beta, \gamma)$ will give the equation of the *axes of the conic*.

275. *To find the equation of a conic circumscribing the triangle of reference.*

The general equation of a conic is

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0.$$

The co-ordinates of the angular points of the triangle are

$$\frac{2\Delta}{a},\ 0,\ 0;\ 0,\ \frac{2\Delta}{b},\ 0;\ \text{and } 0,\ 0,\ \frac{2\Delta}{c}.$$

If these points are on the curve, we must have $u = 0$, $v = 0$, and $w = 0$, as is at once seen by substitution.

Hence the equation of a conic circumscribing the triangle of reference is

$$u'\beta\gamma + v'\gamma\alpha + w'\alpha\beta = 0.$$

276. The condition that a given straight line may touch the conic may be found as in Art. 263, or as follows.

The equation of the lines joining A to the points common to the conic and the straight line (l, m, n), found by eliminating α between the equations of the conic and of the straight line, is

$$lu'\beta\gamma - (v'\gamma + w'\beta)(m\beta + n\gamma) = 0,$$

or $$mw'\beta^2 + nv'\gamma^2 + (mv' + nw' - lu')\beta\gamma = 0.$$

The lines are coincident if (l, m, n) is a tangent; the condition for this is

$$4mnv'w' = (mv' + nw' - lu')^2,$$

which is equivalent to

$$\sqrt{lu'} \pm \sqrt{mv'} \pm \sqrt{nw'} = 0.$$

277. *To find the equation of a conic touching the sides of the triangle of reference.*

The general equation of a conic is

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0.$$

Where the conic cuts $\alpha = 0$, we have

$$v\beta^2 + w\gamma^2 + 2u'\beta\gamma = 0.$$

Hence, if the conic cut $\alpha = 0$ in coincident points we have

$$vw = u'^2, \text{ or } u' = \sqrt{vw}.$$

Similarly, if the conic touch the other sides of the triangle, we have

$$v' = \sqrt{wu}, \text{ and } w' = \sqrt{uv}.$$

Putting λ^2, μ^2, ν^2 instead of u, v, w respectively, we have for the equation,

$$\lambda^2\alpha^2 + \mu^2\beta^2 + \nu^2\gamma^2 \mp 2\mu\nu\beta\gamma \mp 2\nu\lambda\gamma\alpha \mp 2\lambda\mu\alpha\beta = 0.$$

In this equation either one or three of the ambiguous signs must be negative; for otherwise the left side of the equation would be a perfect square, in which case the conic would be two coincident straight lines.

The equation can be written in the form

$$\sqrt{\lambda\alpha} + \sqrt{\mu\beta} + \sqrt{\nu\gamma} = 0.$$

278. *To find the condition that the line $l\alpha + m\beta + n\gamma = 0$ may touch the conic $\sqrt{\lambda\alpha} + \sqrt{\mu\beta} + \sqrt{\nu\gamma} = 0$.*

The condition of tangency can be found as in Art. 276, the result is

$$\frac{\lambda}{l} + \frac{\mu}{m} + \frac{\nu}{n} = 0.$$

279. *To find the equations of the circles which touch the sides of the triangle of reference.*

If D be the point where the inscribed circle touches BC, we know that

$$DC = s - c, \text{ and } DB = s - b.$$

Therefore the equation of AD will be

$$\frac{\beta}{(s-c)\sin C}=\frac{\gamma}{(s-b)\sin B}\ \text{.........(i).}$$

Now the equation of any inscribed conic is

$$\sqrt{\lambda\alpha}+\sqrt{\mu\beta}+\sqrt{\nu\gamma}=0\ \text{............(ii).}$$

The equation of the line joining A to the point of contact of the conic with BC will be given by

$$\sqrt{\mu\beta}+\sqrt{\nu\gamma}=0\,;$$

$$\therefore\ \mu\beta=\nu\gamma\text{.................(iii).}$$

Hence, if (ii) is the inscribed circle, we have from (i) and (iii)

$$\frac{\mu}{b(s-b)}=\frac{\nu}{c(s-c)}.$$

Similarly, by considering the point of contact with CA, we have

$$\frac{\nu}{c(s-c)}=\frac{\lambda}{a(s-a)}.$$

Hence the equation of the inscribed circle is

$$\sqrt{a(s-a)\,\alpha}+\sqrt{b(s-b)\,\beta}+\sqrt{c(s-c)\,\gamma}=0.$$

The equations of the escribed circles can be found in a similar manner.

Ex. 1. Shew that the conic whose equation is

$$\sqrt{a\alpha}+\sqrt{b\beta}+\sqrt{c\gamma}=0,$$

touches the sides of the triangle of reference at their middle points.

Ex. 2. If a conic be inscribed in a triangle, the lines joining the angular points of the triangle to the points of contact with the opposite sides will meet in a point.

280. *To find the equation of a conic which passes through four given points.*

If the diagonal-points of the quadrangle be the angular points of the triangle of reference, the co-ordinates of the four points are given by $\pm f$, $\pm g$, $\pm h$ [Art. 258].

If the four points are on the conic whose equation is

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0,$$

we have the equations

$$uf^2 + vg^2 + wh^2 \pm 2u'gh \pm 2v'hf \pm 2w'fg = 0;$$

$$\therefore u' = v' = w' = 0.$$

Hence the equation of the conic is $u\alpha^2 + v\beta^2 + w\gamma^2 = 0$, with the condition $uf^2 + vg^2 + wh^2 = 0$.

Ex. 1. *Find the locus of the centres of all conics which pass through four given points.*

Let the four points be $\pm f, \pm g, \pm h$.

The equation of any conic will be

$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0,$$

with the condition

$$uf^2 + vg^2 + wh^2 = 0 \text{..........................} \text{(i)}.$$

The co-ordinates of the centre of the conic are given by

$$\frac{u\alpha}{a} = \frac{v\beta}{b} = \frac{w\gamma}{c}.$$

Substitute for u, v, w in (i), and we have the equation of the locus, viz.

$$\frac{af^2}{\alpha} + \frac{bg^2}{\beta} + \frac{ch^2}{\gamma} = 0. \quad \text{[See Art. 209.]}$$

Ex. 2. The polars of a given point with respect to a system of conics passing through four given points will pass through a fixed point.

281. *To find the equation of a conic touching four given straight lines.*

Let the triangle formed by the diagonals of the quadrilateral be taken for the triangle of reference, then [Art 259] the equations of the four lines will be of the form

$$l\alpha \pm m\beta \pm n\gamma = 0.$$

The conic whose equation is

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0 \text{ ...(i)}$$

will touch the line (l, m, n) if

$$Ul^2 + Vm^2 + Wn^2 + 2U'mn + 2V'nl + 2W'lm = 0.$$

If therefore the conic touch all four of the lines, we must have

$$U' = V' = W' = 0,$$

that is

$$v'w' - uu' = 0,$$

$$w'u' - vv' = 0,$$

$$u'v' - ww' = 0;$$

$$\therefore u' = v' = w' = 0,$$

or else (i) is a perfect square, and the conic a pair of coincident straight lines.

Hence we must have $u' = v' = w' = 0$, and the condition of tangency is $\quad l^2vw + m^2wu + n^2uv = 0.$

Hence every conic touching the four straight lines is included in the equation

$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0,$$

with the condition

$$\frac{l^2}{u} + \frac{m^2}{v} + \frac{n^2}{w} = 0.$$

Ex. 1. *Find the locus of the centres of the conics which touch four given straight lines.*

Any conic is given by

$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0,$$

with the condition

$$\frac{l^2}{u} + \frac{m^2}{v} + \frac{n^2}{w} = 0.$$

The co-ordinates of the centre of the conic are given by

$$\frac{u\alpha}{a} = \frac{v\beta}{b} = \frac{w\gamma}{c};$$

therefore the equation of the locus of the centres is the straight line

$$\frac{l^2\alpha}{a} + \frac{m^2\beta}{b} + \frac{n^2\gamma}{c} = 0.$$

This straight line goes through the middle points of the three diagonals of the quadrilateral. [See Art. 217.]

Ex. 2. The locus of the pole of a given line with respect to a system of conics inscribed in the same quadrilateral is a straight line.

Ex. 3. Shew that, if the conic $u\alpha^2 + v\beta^2 + w\gamma^2 = 0$ be a parabola, it will touch the four lines given by $a\alpha \pm b\beta \pm c\gamma = 0$.

282. When the equation of a conic is of the form $u\alpha^2 + v\beta^2 + w\gamma^2 = 0$, each angular point of the triangle of reference is the pole of the opposite side. This is at once seen if the co-ordinates of an angular point of the triangle be substituted in the equation of the polar of $(\alpha', \beta', \gamma')$, viz. $\quad u\alpha'\alpha + v\beta'\beta + w\gamma'\gamma = 0.$

Conversely, if the triangle of reference be self-polar, the equation of the conic will be of the form $u\alpha^2 + v\beta^2 + w\gamma^2 = 0$. For, the equation of the polar of $A\left(\frac{2\Delta}{a}, 0, 0\right)$, with respect to the conic given by the general equation, is

$$u\alpha + w'\beta + v'\gamma = 0.$$

Hence, if BC be the polar of A, we have $w' = v' = 0$. Similarly, if CA be the polar of B, we have $w' = u' = 0$. Hence u', v', w' are all zero.

283. If two conics intersect in four real points, and we take the diagonal-points of the quadrangle formed by the four points for the triangle of reference, the equations of the two conics will [Art. 280] be of the form

$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0, \text{ and } u'\alpha^2 + v'\beta^2 + w'\gamma^2 = 0.$$

So that, as we have seen in Art. 213, any two conics which intersect in four real points have a common self-polar triangle.

When two of the points of intersection of two conics are real and the other two imaginary, two angular points of the common self-polar triangle are imaginary. When all four points of intersection of two conics are imaginary, they will have a real self-polar triangle. [See Ferrers' Trilinears, or Salmon's Conic Sections, Art. 282.]

284. If two tangents and their chord of contact be the sides of the triangle of reference, the equation of the conic will be of the form

$$\alpha^2 = 2\kappa\beta\gamma \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

for (i) is a conic which has double contact with the conic $\beta\gamma = 0$, the chord of contact being $\alpha = 0$. [See Art. 187.]

285. *To find the equation of the circle with respect to which the triangle of reference is self-polar.*

The equations of all conics with respect to which the triangle of reference is self-polar are of the form

$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0.$$

The equation of any circle can be written in the form

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + (\lambda\alpha + \mu\beta + \nu\gamma)(a\alpha + b\beta + c\gamma) = 0.$$

If these equations represent the same curve, we have

$$u = \lambda a, \quad v = \mu b, \quad w = \nu c,$$

$$a + \mu c + \nu b = 0, \; b + \nu a + \lambda c = 0, \text{ and } c + \lambda b + \mu a = 0.$$

Whence $\lambda = -\cos A$, $\mu = -\cos B$, and $\nu = -\cos C$.

The required equation is therefore

$$a \cos A \,.\, \alpha^2 + b \cos B \,.\, \beta^2 + c \cos C \,.\, \gamma^2 = 0.$$

286. *To find the equation of the nine-point circle.*

Let the equation of the circle be

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta - (\lambda\alpha + \mu\beta + \nu\gamma)(a\alpha + b\beta + c\gamma) = 0.$$

This circle cuts $\alpha = 0$ where $b\beta = c\gamma$;

$$\therefore abc - 2(\mu c + \nu b)\, bc = 0,$$

or
$$\frac{\mu}{b} + \frac{\nu}{c} = \frac{a^2}{2abc}.$$

Similarly
$$\frac{\nu}{c} + \frac{\lambda}{a} = \frac{b^2}{2abc},$$

and
$$\frac{\lambda}{a} + \frac{\mu}{b} = \frac{c^2}{2abc}.$$

Hence $\qquad 2\lambda = \cos A$, $2\mu = \cos B$, and $2\nu = \cos C$;

therefore the equation of the circle is

$$2a\beta\gamma + 2b\gamma\alpha + 2c\alpha\beta$$
$$- (\alpha \cos A + \beta \cos B + \gamma \cos C)(a\alpha + b\beta + c\gamma) = 0,$$

or $\quad a\beta\gamma + b\gamma\alpha + c\alpha\beta - \alpha^2 a \cos A - \beta^2 b \cos B - \gamma^2 c \cos C = 0.$

The form of this equation shews that the nine-point circle, the circumscribed circle, and the self-conjugate circle have a common radical axis, the equation of the radical axis being

$$\alpha \cos A + \beta \cos B + \gamma \cos C = 0.$$

Ex. 1. The centre of the self-conjugate circle of a triangle is its orthocentre.

Ex. 2. The locus of the centres of all rectangular hyperbolas described about a given triangle is the nine-point circle.

287. *Pascal's Theorem. If a hexagon be inscribed in a conic, the three points of intersection of the three pairs of opposite sides will be on a straight line.*

Let the angular points of the hexagon be A, F, B, D, C, E. Take ABC for the triangle of reference, and let the points D, E, F be $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$, and $(\alpha''', \beta''', \gamma''')$.

Let the equation of the conic be

$$\frac{\lambda}{\alpha}+\frac{\mu}{\beta}+\frac{\nu}{\gamma}=0 \text{ ....(i).}$$

The equations of BD and AE will be $\frac{\alpha}{\alpha'}=\frac{\gamma}{\gamma'}$, and $\frac{\beta}{\beta''}=\frac{\gamma}{\gamma''}$; therefore at their intersection, $\frac{\alpha}{\frac{\alpha'}{\gamma'}}=\frac{\beta}{\frac{\beta''}{\gamma''}}=\frac{\gamma}{1}$.

Similarly CD, AF meet in the point $\left(\frac{\alpha'}{\beta'},\ 1,\ \frac{\gamma'''}{\beta'''}\right)$.

And CE, BE meet in the point $\left(1, \frac{\beta''}{\alpha''},\ \frac{\gamma'''}{\alpha'''}\right)$.

The three points will lie on a straight line if

$$\begin{vmatrix} \frac{\alpha'}{\gamma'}, & \frac{\beta''}{\gamma''}, & 1 \\ \frac{\alpha'}{\beta'}, & 1, & \frac{\gamma'''}{\beta'''} \\ 1, & \frac{\beta''}{\alpha''}, & \frac{\gamma'''}{\alpha'''} \end{vmatrix}=0, \text{ or if } \begin{vmatrix} \frac{1}{\gamma'}, & \frac{1}{\gamma''}, & \frac{1}{\gamma'''} \\ \frac{1}{\beta'}, & \frac{1}{\beta''}, & \frac{1}{\beta'''} \\ \frac{1}{\alpha'}, & \frac{1}{\alpha''}, & \frac{1}{\alpha'''} \end{vmatrix}=0 \text{ ......(ii).}$$

But, since the three points D, E, F are on the conic (i), we have

$$\frac{\lambda}{\alpha'}+\frac{\mu}{\beta'}+\frac{\nu}{\gamma'}=0,$$

$$\frac{\lambda}{\alpha''}+\frac{\mu}{\beta''}+\frac{\nu}{\gamma''}=0,$$

and $$\frac{\lambda}{\alpha'''}+\frac{\mu}{\beta'''}+\frac{\nu}{\gamma'''}=0.$$

By the elimination of λ, μ, ν we see that the condition (ii) is satisfied, which proves the proposition. [See also Art. 319, Ex. 3.]

Since six points can be taken in order in sixty different ways, there are sixty hexagons corresponding to six points on a conic; and, since Pascal's Theorem is true for every one of these hexagons, there are sixty Pascal lines corresponding to six points on a conic.

288. If a hexagon circumscribe a conic, the points of contact of its sides will be the angular points of a hexagon inscribed in the conic. Each angular point of the circumscribed hexagon will be the pole of the corresponding side of the inscribed hexagon; therefore a diagonal of the circumscribing hexagon, that is a line joining a pair of its opposite angular points, will be the polar of the point of intersection of a pair of opposite sides of the inscribed hexagon. But the three points of intersection of pairs of opposite sides of the inscribed hexagon lie on a straight line by Pascal's Theorem; hence their three polars, that is the three diagonals of the circumscribing hexagon, will meet in a point. This proves Brianchon's Theorem:—*if a hexagon be described about a conic, the three diagonals will meet in a point.*

289. If we are given *five* tangents to a conic we can find their points of contact by Brianchon's Theorem. For, let A, B, C, D, E be the angular points of a pentagon formed by the five given tangents; then, if K be the point of contact of AB, A, K, B, C, D, E are the angular points of a circumscribing *hexagon*, two sides of which are coincident. By Brianchon's Theorem, DK passes through the point of intersection of AC and BE; hence K is found. The other points of contact can be found in a similar manner.

Similarly, by means of Pascal's Theorem, we can find the tangents to a conic at five given points. For, let A, B, C, D, E be the five given points, and let F be the point on the conic indefinitely near to A; then, by Pascal's Theorem, the three points of intersection of AB and DE; of BC and EF; and of CD and FA lie on a straight line. Hence, if the line joining the point of intersection of AB and DE to the point of intersection of BC and EA meet CD in H, AH will be the tangent at A. The other tangents can be found in a similar manner.

AREAL CO-ORDINATES.

290. The position of any point P is determined if the ratios of the triangles PBC, PCA, PAB to the triangle of reference ABC be given. These ratios are denoted by x, y, z respectively, and are called the *areal* co-ordinates of the point P.

The areal co-ordinates of any point are connected by the relation
$$x+y+z=1.$$

Since $x=\frac{a\alpha}{2\Delta}$, $y=\frac{b\beta}{2\Delta}$, and $z=\frac{c\gamma}{2\Delta}$, we at once find the equation in areal co-ordinates which corresponds to any given homogeneous equation in trilinear co-ordinates, by substituting in the given equation $\frac{x}{a}$, $\frac{y}{b}$, $\frac{z}{c}$ for α, β, γ respectively; for example the equation of the line at infinity is $x+y+z=0$. We will however find the areal equation of the circumscribing circle independently.

291. *To find the equation in areal co-ordinates of the circle which circumscribes the triangle of reference.*

If P be any point on the circle circumscribing the triangle ABC, then by Ptolemy's Theorem (Euclid VI. D.) we have
$$PA\,.\,BC \pm PB\,.\,CA \pm PC\,.\,AB=0 \;\ldots\ldots\ldots\text{(i)}.$$

But since the angles BPC and BAC are equal, we have $\frac{PB\,.\,PC}{AB\,.\,AC}=x$, and similarly for y and z; hence, paying regard to the signs of x, y, z, we have from (i)
$$a\,.\,\frac{PA\,.\,PB\,.\,PC}{bcx}+b\,.\,\frac{PA\,.\,PB\,.\,PC}{cay}+c\,.\,\frac{PA\,.\,PB\,.\,PC}{abz}=0,$$
or
$$\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}=0,$$
which is the equation required.

292. If the conic represented by the general equation of the second degree in trilinear co-ordinates, viz.

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0,$$

be the same as that represented in areal co-ordinates by the equation

$$\lambda x^2 + \mu y^2 + \nu z^2 + 2\lambda' yz + 2\mu' zx + 2\nu' xy = 0;$$

then, since $\dfrac{x}{a\alpha} = \dfrac{y}{b\beta} = \dfrac{z}{c\gamma}$, we have

$$\frac{u}{\lambda a^2} = \frac{v}{\mu b^2} = \frac{w}{\nu c^2} = \frac{u'}{\lambda' bc} = \frac{v'}{\mu' ca} = \frac{w'}{\nu' ab}.$$

Hence we can obtain the relation between the coefficients in the areal equation which corresponds to any given relation between the coefficients in the trilinear equation.

For example, the condition that $u\alpha^2 + v\beta^2 + w\gamma^2 = 0$ may be a rectangular hyperbola is $u + v + w = 0$; hence the condition that $\lambda x^2 + \mu y^2 + \nu z^2 = 0$ may be a rectangular hyperbola is $$\lambda a^2 + \mu b^2 + \nu c^2 = 0.$$

TANGENTIAL CO-ORDINATES.

293. If l, m, n be the three constants in the trilinear or areal equation of any straight line, the position of the line will be determined when l, m and n are given; and by changing the values of l, m, and n the equation may be made to represent any straight line whatever.

The quantities l, m, and n which thus define the position of a straight line are called the *co-ordinates* of the *line*.

If the equation of a straight line in areal co-ordinates be $$lx + my + nz = 0,$$

the lengths of the perpendiculars on the line from the angular points of the triangle of reference will be proportional to l, m, n. This follows at once from Art. 257; we will however give an independent proof.

Let the lengths of the perpendiculars from the angular points A, B, C of the triangle of reference be p, q, r

respectively. Let the line cut BC in the point K, and let the co-ordinates of K be $0, y', z'$.

Then $$q : r :: BK : CK :: -z' : y'.$$

But, since K is on the line, $my' + nz' = 0$; therefore

$$q : r :: m : n.$$

294. The lengths of the perpendiculars on a straight line from the angular points of the triangle of reference may be called the co-ordinates of the line. If any two of these perpendiculars be drawn in different directions they must be considered to have different signs.

From the preceding Article we see that the equation of a line whose co-ordinates are p, q, r is $px + qy + rz = 0$.

When the lengths of two of the perpendiculars on a straight line are given, there are two and only two positions of the line; so that, when two of the co-ordinates of the line are given, the third has one of two particular values. Hence there must be some identical relation connecting the three co-ordinates of a line, and that relation must be of the second degree.

295. *To find the identical relation which exists between the co-ordinates of any line.*

Let θ be the angle the line makes with BA, then we have $q - p = c \sin \theta$, and $q - r = a \sin(\theta + B)$. The elimination of θ gives the required relation, viz.

$$a^2 (q-p)^2 - 2ac \cos B (q-p)(q-r) + c^2 (q-r)^2 = 4\Delta^2,$$

or

$$a^2 (p-q)(p-r) + b^2 (q-r)(q-p) + c^2 (r-p)(r-q) = 4\Delta^2.$$

296. If the line $px + qy + rz = 0$ pass through a fixed point (f, g, h), then

$$pf + qg + rh = 0 \text{..................} \text{(i)}.$$

So that the co-ordinates of all the lines which pass through the point whose areal co-ordinates are f, g, h satisfy the relation (i).

Hence the *equation* of a *point* is of the first degree.

297. If the co-ordinates of a straight line are connected by any relation the line will envelope a curve, and the equation which expresses that relation is called the *tangential equation* of the curve.

We have seen that the tangential equation of a conic is of the second degree, and that every curve whose equation is of the second degree is a conic. If $\psi(l, m, n)=0$ be the tangential equation of the conic whose areal equation is $\phi(x, y, z)=0$, and if the coefficients in the equation $\phi=0$ be u, v, w, u', v', w'; the corresponding coefficients in the equation $\psi=0$ will be U, V, W, U', V', W', the minors of u, v, w, u', v', w' respectively in the determinant

$$\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}.$$

Since u, v, w, u', v', w' are proportional to the minors of U, V, W, U', V', W' in the determinant

$$\begin{vmatrix} U, & W', & V' \\ W', & V, & U' \\ V', & U', & W \end{vmatrix}; \text{ [See Art. 239]}$$

it follows that if $\psi(l, m, n)=0$ be the tangential equation of the conic whose areal equation is $\phi(x, y, z)=0$, then $\phi(l, m, n)=0$ will be the tangential equation of the conic whose areal equation is $\psi(x, y, z)=0$.

298. We can find the equation of the point of contact of any tangent by an investigation similar to that in Art. 261.

The equation is

$$p\frac{d\phi}{dp'}+q\frac{d\phi}{dq'}+r\frac{d\phi}{dr'}=0,$$

or

$$p'\frac{d\phi}{dp}+q'\frac{d\phi}{dq}+r'\frac{d\phi}{dr}=0,$$

where $\phi(p, q, r)$ is the equation of the conic, and p', q', r' are the co-ordinates of the tangent.

If (p', q', r') be not a tangent to the curve, the above equation will be the equation of the pole of (p', q', r').

The centre is the pole of the line at infinity whose co-ordinates are 1, 1, 1; hence the equation of the centre of the curve is $$\frac{d\phi}{dp}+\frac{d\phi}{dq}+\frac{d\phi}{dr}=0.$$

299. We shall conclude this chapter by the solution of some examples.

(1) *If the sides of two triangles touch a given conic, their six angular points will lie on another conic.*

Take one of the triangles for the triangle of reference.

Let the equation of the given conic be

$$\sqrt{\lambda\alpha}+\sqrt{\mu\beta}+\sqrt{\nu\gamma}=0.$$

Let the equations of the sides of the second triangle be

$$l_1\alpha+m_1\beta+n_1\gamma=0,\ \ l_2\alpha+m_2\beta+n_2\gamma=0,$$

and $$l_3\alpha+m_3\beta+n_3\gamma=0.$$

Then

$$L\,(l_2\alpha+m_2\beta+n_2\gamma)\,(l_3\alpha+m_3\beta+n_3\gamma)+M\,(l_3\alpha+m_3\beta+n_3\gamma)\,(l_1\alpha+m_1\beta+n_1\gamma)$$
$$+N\,(l_1\alpha+m_1\beta+n_1\gamma)\,(l_2\alpha+m_2\beta+n_2\gamma)=0$$

will be the general equation of a conic circumscribing the triangle formed by these straight lines.

This conic will pass through the angular points of the triangle of reference if the coefficients of α^2, β^2 and γ^2 are all zero. That is, if

$$L\ \ l_2l_3+Ml_3l_1+Nl_1l_2=0,$$
$$Lm_2m_3+Mm_3m_1+Nm_1m_2=0,$$

and $$Ln_2n_3+Mn_3n_1+Nn_1n_2=0.$$

Eliminating L, M, N, we see that the condition to be satisfied is

$$\begin{vmatrix} l_2l_3, & l_3l_1, & l_1l_2 \\ m_2m_3, & m_3m_1, & m_1m_2 \\ n_2n_3, & n_3n_1, & n_1n_2 \end{vmatrix}=0,\ \text{or}\ \begin{vmatrix} \frac{1}{l_1}, & \frac{1}{l_2}, & \frac{1}{l_3} \\ \frac{1}{m_1}, & \frac{1}{m_2}, & \frac{1}{m_3} \\ \frac{1}{n_1}, & \frac{1}{n_2}, & \frac{1}{n_3} \end{vmatrix}=0.$$

But, since the three lines touch the given conic, we have

$$\frac{\lambda}{l_1}+\frac{\mu}{m_1}+\frac{\nu}{n_1}=0,\ \frac{\lambda}{l_2}+\frac{\mu}{m_2}+\frac{\nu}{n_2}=0,\ \text{and}\ \frac{\lambda}{l_3}+\frac{\mu}{m_3}+\frac{\nu}{n_3}=0.$$

Eliminating λ, μ, ν, we see that the required condition is satisfied.

[See also Art. 322, Ex. 2.]

(2) *If one triangle can be inscribed in one conic with its sides touching another conic, then an infinite number of triangles can be so described.*

Let ABC be the triangle whose angular points are on the conic Σ, and whose sides touch the conic S.

Let any other tangent to S be drawn cutting Σ in the points B', C', and let the other tangents to S through B', C' meet at A'. Then $A'B'C'$ and ABC are two triangles whose sides touch S. Therefore by the preceding question the six points A, B, C, B', C', A' are on a conic. But five of the points, viz. A, B, C, B', C', are on the conic Σ, and only one conic will pass through five points, therefore C' also is on Σ.

(3) *Four circles are described so that each of the four triangles, formed by each three of four given straight lines, is self-polar with respect to one of the circles; prove that these four circles and the circle circumscribing the triangle formed by the diagonals of the quadrilateral have a common radical axis.*

Take the triangle formed by the diagonals for the triangle of reference, then the equations of the four straight lines will be $l\alpha \pm m\beta \pm n\gamma = 0$. All conics with respect to which the lines

$$l\alpha + m\beta + n\gamma = 0,\ \ l\alpha - m\beta + n\gamma = 0, \text{ and } l\alpha + m\beta - n\gamma = 0$$

form a self-polar triangle are included in the equation

$$L\,(l\alpha + m\beta + n\gamma)^2 + M\,(l\alpha - m\beta + n\gamma)^2 + N\,(l\alpha + m\beta - n\gamma)^2 = 0 \ldots\ldots \text{(i)}.$$

If this conic be a circle its equation can be put in the form

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + (\lambda\alpha + \mu\beta + \nu\gamma)(a\alpha + b\beta + c\gamma) = 0 \ldots\ldots\ldots \text{(ii)},$$

and $\lambda\alpha + \mu\beta + \nu\gamma = 0$ is the radical axis of (ii) and of the circumscribing circle. Comparing coefficients of α^2, β^2 and γ^2 in (i) and (ii) we obtain, for the equation of the radical axis

$$\frac{l^2}{a}\alpha + \frac{m^2}{b}\beta + \frac{n^2}{c}\gamma = 0.$$

This is clearly the same for all four circles.

(4) *A line cuts two given conics in P, P', and Q, Q', so that the range P, Q, P', Q' is harmonic; find the envelope of the line.*

Refer the conics to their common self-conjugate triangle and let their equations be

$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0, \quad u'\alpha^2 + v'\beta^2 + w'\gamma^2 = 0.$$

Let the equation of the line be

$$l\alpha + m\beta + n\gamma = 0.$$

Then the lines AP, AP' are given by the equation

$$u\,(m\beta + n\gamma)^2 + l^2 v\beta^2 + l^2 w\gamma^2 = 0,$$

or
$$\beta^2\,(um^2 + vl^2) + 2umn\beta\gamma + (un^2 + wl^2)\,\gamma^2 = 0.$$

And similarly AQ, AQ' are given by

$$\beta^2(u'm^2+v'l^2)+2u'mn\beta\gamma+(u'n^2+w'l^2)\gamma^2=0.$$

If therefore $A\{PQP'Q'\}$ is harmonic, we must have [Art. 58]

$$(um^2+vl^2)(u'n^2+w'l^2)+(un^2+wl^2)(u'm^2+v'l^2)=2uu'm^2n^2;$$

which reduces to

$$(vw'+wv')l^2+(wu'+uw')m^2+(uv'+vu')n^2=0.$$

This condition shews that the line always touches the conic

$$\frac{\alpha^2}{vw'+wv'}+\frac{\beta^2}{wu'+uw'}+\frac{\gamma^2}{uv'+vu'}=0.$$

It is easy to shew that the envelope touches the eight tangents to the given conics at their four points of intersection.

(5) *The director-circles of all conics which are inscribed in the same quadrilateral have a common radical axis.*

Let the triangle formed by the diagonals of the quadrilateral be taken for the triangle of reference.

Then the equations of the four lines will be $l\alpha\pm m\beta\pm n\gamma=0$. [Art. 259.]

The equation of any one of the conics will be $u\alpha^2+v\beta^2+w\gamma^2=0$. [Art. 281.]

The equation of the two tangents from the point $(\alpha'\beta'\gamma')$ is

$$(u\alpha^2+v\beta^2+w\gamma^2)(u\alpha'^2+v\beta'^2+w\gamma'^2)-(u\alpha'\alpha+v\beta'\beta+w\gamma'\gamma)^2=0.$$

The condition that these lines may be perpendicular is [Art. 268]

$$u(v\beta'^2+w\gamma'^2)+v(w\gamma'^2+u\alpha'^2)+w(u\alpha'^2+v\beta'^2)+2vw\beta'\gamma'\cos A$$
$$+2wu\gamma'\alpha'\cos B+2uv\alpha'\beta'\cos C=0.$$

Hence the equation of the director-circle of the conic $u\alpha^2+v\beta^2+w\gamma^2=0$ will be

$$\frac{\beta^2+\gamma^2+2\beta\gamma\cos A}{u}+\frac{\gamma^2+\alpha^2+2\gamma\alpha\cos B}{v}+\frac{\alpha^2+\beta^2+2\alpha\beta\cos C}{w}=0\ldots\text{(i)}.$$

But, since the conic touches the four lines $l\alpha\pm m\beta\pm n\gamma=0$, we have

$$\frac{l^2}{u}+\frac{m^2}{v}+\frac{n^2}{w}=0\ldots\ldots\ldots\ldots\text{(ii)}.$$

Comparing (i) and (ii) we see that all the director-circles pass through the points given by

$$\frac{\beta^2+\gamma^2+2\beta\gamma\cos A}{l^2}=\frac{\gamma^2+\alpha^2+2\gamma\alpha\cos B}{m^2}=\frac{\alpha^2+\beta^2+2\alpha\beta\cos C}{n^2}.$$

[See also Art. 243, Ex. (2), and Art. 307.]

(6) *To find the tangential equation of the circle with respect to which the triangle of reference is self-polar.*

The trilinear equation of the circle is

$$\alpha^2a\cos A+\beta^2b\cos B+\gamma^2c\cos C=0.$$

The line $l\alpha + m\beta + n\gamma = 0$ will touch the circle, if

$$\frac{l^2}{a \cos A} + \frac{m^2}{b \cos B} + \frac{n^2}{c \cos C} = 0.$$

If p, q, r be the perpendiculars on the line from the angular points of the triangle

$$\frac{p}{\frac{l}{a}} = \frac{q}{\frac{m}{b}} = \frac{r}{\frac{n}{c}} \text{ [Art. 257].}$$

Hence from the condition of tangency

$$p^2 \tan A + q^2 \tan B + r^2 \tan C = 0,$$

which is the required tangential equation.

Examples on Chapter XIII.

1. Shew that the minor axis of an ellipse inscribed in a given triangle cannot exceed the diameter of the inscribed circle.

2. Find the area of a triangle in terms of the trilinear or areal co-ordinates of its angular points.

3. If four conics have a common self-conjugate triangle, the four points of intersection of any two and the four points of intersection of the other two lie on a conic.

4. Shew that the eight points of contact of two conics with their common tangents lie on a conic.

5. Shew that the eight tangents to two conics at their common points touch a conic.

6. Any three pairs of points which divide the three diagonals of a quadrilateral harmonically are on a conic.

7. Find the equation of the nine-point circle by considering it as the circle circumscribing the triangle formed by the lines

$$a\alpha - b\beta - c\gamma = 0,\ b\beta - c\gamma - a\alpha = 0, \text{ and } c\gamma - a\alpha - b\beta = 0.$$

8. Shew that the equation of the circle concentric with $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$ and of radius r is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta - \frac{r^2 - R^2}{abc}(a\alpha + b\beta + c\gamma)^2 = 0,$$

where R is the radius of the circle circumscribing the triangle of reference.

9. The equation of the circumscribing conic, whose diameters parallel to the sides of the triangle of reference are r_1, r_2, r_3 is

$$\frac{a}{r_1^2\alpha}+\frac{b}{r_2^2\beta}+\frac{c}{r_3^2\gamma}=0.$$

10. *ABC* is a triangle inscribed in a conic, and the tangents to the conic at *A*, *B*, *C* are *B′C′*, *C′A′*, *A′B′* respectively; shew that *AA′*, *BB′*, and *CC′* meet in a point. Shew also that, if *D* be the point of intersection of *BC*, *B′C′*; *E* the point of intersection of *CA*, *C′A′*, and *F* the point of intersection of *AB*, *A′B′*; *D*, *E*, *F* will be a straight line.

11. Lines are drawn from the angular points *A*, *B*, *C* of a triangle through a point *P* to meet the opposite sides in *A′*, *B′*, *C′*. *B′C′* meets *BC* in *K*, *C′A′* meets *CA* in *L*, and *A′B′* meets *AB* in *M*. Shew that *K*, *L*, *M* are on a straight line. Shew also (i) that if *P* moves on a fixed straight line then *KLM* will touch a conic inscribed in the triangle *ABC*; (ii) that if *P* moves on a fixed conic circumscribing the triangle *ABC*, then *KLM* will pass through a fixed point; (iii) that if *P* moves on a fixed conic touching two sides of the triangle where they are met by the third, *KLM* will envelope a conic.

12. Lines drawn through the angular points *A*, *B*, *C* of a triangle and through a point *O* meet the opposite sides in *A′*, *B′*, *C′*; and those drawn through a point *O′* meet the opposite sides in *A″*, *B″*, *C″*. If *P* be the point of intersection of *B′C′* and *B″C″*, *Q* be the point of intersection of *C′A′*, *C″A″*, and *R* be the point of intersection of *A′B′*, *A″B″*; shew that *AP*, *BQ*, *CR* will meet in some point *Z*. Shew also that, if *O*, *O′* be any two points on a fixed conic through *A*, *B*, *C*, the point *Z* will be fixed.

13. The locus of the pole of a given straight line with respect to a system of conics through four given points is a conic which passes through the diagonal-points of the quadrangle formed by the given points.

14. The envelope of the polar of a given point with respect to a system of conics touching four given straight lines is a conic which touches the diagonals of the quadrilateral formed by the given lines.

15. Shew that the locus of the points of contact of tangents, drawn parallel to a fixed line, to the conics inscribed in a given quadrilateral, is a cubic; and notice any remarkable points, connected with the quadrilateral, through which the cubic passes.

16. An ellipse is inscribed within a triangle and has its centre at the centre of the circumscribing circle. Shew that its major and minor axes are $R+d$ and $R-d$ respectively, R being the radius of the circumscribing circle and d the distance between the centre and the orthocentre.

17. Prove that a conic circumscribing a triangle ABC will be an ellipse if the centre lie within the triangle DEF or within the angles vertically opposite to the angles of the triangle DEF, where D, E, F are the middle points of the sides of the triangle ABC.

18. Shew that the locus of the foci of parabolas to which the triangle of reference is self-polar is the nine-point circle.

19. Shew that the locus of the foci of all conics touching the four lines $l\alpha \pm m\beta \pm n\gamma = 0$ is the cubic

$$\frac{P_1^2}{l\alpha + m\beta + n\gamma} + \frac{P_2^2}{l\alpha - m\beta - n\gamma} + \frac{P_3^2}{-l\alpha + m\beta - n\gamma} + \frac{P_4^2}{-l\alpha - m\beta + n\gamma} = 0,$$

where $P_1^2 = l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C$, and P_2^2, P_3^2, P_4^2 have similar values.

20. If a conic be inscribed in a given triangle, and its major axis pass through the fixed point (f, g, h), the locus of its focus is the cubic

$$f\alpha(\beta^2 - \gamma^2) + g\beta(\gamma^2 - \alpha^2) + h\gamma(\alpha^2 - \beta^2) = 0.$$

21. If the centre of a conic inscribed in a triangle move along a fixed straight line, the foci will lie on a cubic circumscribing the triangle.

22. The locus of the centres of the rectangular hyperbolas with respect to which the triangle of reference is self-conjugate is the circumscribing circle.

23. The locus of the centres of all rectangular hyperbolas inscribed in the triangle of reference is the self-conjugate circle.

24. Shew that the nine-point circle of a triangle touches the inscribed circle and each of the escribed circles.

25. The tangents to the nine-point circle at the points where it touches the inscribed and escribed circles form a quadrilateral, each diagonal of which passes through an angular point of the triangle, and the lines joining corresponding angular points of the original triangle and of the triangle formed by the diagonals are all parallel to the radical axis of the nine-point circle and the circumscribing circle.

26. The polars of the points A, B, C with respect to a conic are $B'C'$, $C'A'$, $A'B'$ respectively; shew that AA', BB', CC' meet in a point.

27. If an equilateral hyperbola pass through the middle points of the sides of a triangle ABC and cuts the sides BC, CA, AB again in α, β, γ respectively, then $A\alpha$, $B\beta$, $C\gamma$ meet in a point on the circumscribed circle of the triangle ABC.

28. Shew that the locus of the intersection of the polars of all points in a given straight line with respect to two given conics is a conic circumscribing their common self-conjugate triangle.

29. Two conics have double contact; shew that the locus of the poles with respect to one conic of the tangents to the other is a conic which has double contact with both at their common points.

30. Two triangles are inscribed in a conic; shew that their six sides touch another conic.

31. Two triangles are self-polar with respect to a conic; shew that their six angular points are on a second conic, and that their six sides touch a third conic.

32. If one triangle can be described self-polar to a given conic and with its angular points on another given conic, an infinite number of triangles can be so described.

33. A system of similar conics have a common self-conjugate triangle; shew that their centres are on a curve of the 4th degree which passes through the circular points at infinity and of which the angular points of the triangle are double points.

34. If A, B, C, A', B', C' be six points such that AA', BB', CC' meet in a point, then will the six straight lines AB', AC', BC', BA', CA' and CB' touch a conic.

35. A conic is inscribed in a triangle and is such that the normals at the points of contact meet in a point; prove that the point of concurrence describes a cubic curve whose asymptotes are perpendicular to the sides of the triangle.

36. If p_1, p_2, p_3, p_4 be the lengths of the perpendiculars drawn from the vertices A, B, C, D of a quadrilateral circumscribed about a conic on any other tangent to the conic, shew that the ratio of $p_1 p_3$ to $p_2 p_4$ will be constant.

37. The polars with respect to any conic of the angular points A, B, C of a triangle meet the opposite sides in A', B', C'; shew that the circles on AA', BB', CC' as diameters have a common radical axis.

38. A parabola touches one side of a triangle in its middle point, and the other two sides produced; prove that the perpendiculars drawn from the angular points of the triangle upon any tangent to the parabola are in harmonical progression.

39. Shew that the tangential equation of the circumscribing circle is $a\sqrt{p} + b\sqrt{q} + c\sqrt{r} = 0$. Hence shew that the tangential equation of the nine-point circle is

$$a\sqrt{(q+r)} + b\sqrt{(r+p)} + c\sqrt{(p+q)}.$$

40. The locus of the centre of a conic inscribed in a given triangle, and having the sum of the squares of its axis constant, is a circle.

41. The director circles of all conics inscribed in the same triangle are cut orthogonally by the circle to which the triangle of reference is self-polar.

42. The circles described on the diagonals of a complete quadrilateral are cut orthogonally by the circle round the triangle formed by the diagonals.

43. If three conics circumscribe the same quadrilateral, shew that a common tangent to any two is cut harmonically by the third.

44. If three conics are inscribed in the same quadrilateral the tangents to two of them at a common point and the tangents to the third from that point form a harmonic pencil.

45. The locus of a point the pairs of tangents from which to two given conics form a harmonic pencil is a conic on which lie the eight points in which the given conics touch their common tangents.

46. The locus of a point from which the tangents drawn to two equal circles form a harmonic pencil is a conic, which is an ellipse if the circles cut at an angle less than a right angle, and two parallel straight lines if they cut at right angles.

47. A triangle is circumscribed about one conic and two of its angular points are on a second conic; find the locus of the third angular point.

48. A triangle is inscribed in one conic and two of its sides touch a second conic; find the envelope of the third side.

49. The angular points of a triangle are on the sides of a given triangle, and two of its sides pass through fixed points; shew that the third side will envelope a conic.

50. From the angular points of the fundamental triangle pairs of tangents are drawn to $(uvwu'v'w' \between xyz)^2 = 0$, and each pair determine with the opposite sides a pair of points. Find the equation to the conic on which these six points lie, and shew that the conic

$$\sqrt{x(v'w' - uu')} + \sqrt{y(w'u' - vv')} + \sqrt{z(u'v' - ww')} = 0$$

and the above two conics have a common inscribed quadrilateral.

CHAPTER XIV.

Reciprocal Polars. Projections.

300. If we have any figure consisting of any number of points and straight lines in a plane, and we take the polars of those points and the poles of the lines, with respect to a fixed conic C, we obtain another figure which is called the *polar reciprocal* of the former with respect to the *auxiliary conic* C.

When a point in one figure and a line in the reciprocal figure are pole and polar with respect to the auxiliary conic C, we shall say that they *correspond* to one another.

If in one figure we have a curve S the lines which correspond to the different points of S will all touch some curve S'. Let the lines corresponding to the two points P, Q of S meet in T; then T is the pole of the line PQ with respect to C, that is the line PQ corresponds to the point T. Now, if the point Q move up to and ultimately coincide with P, the two corresponding tangents to S' will also ultimately coincide with one another, and their point of intersection T will ultimately be on the curve S'. So that a tangent to the curve S corresponds to a point on the curve S', just as a tangent to S' corresponds to a point on S. Hence we see that S is generated from S' exactly as S' is from S.

301. If any line L cut the curve S in any number of points P, Q, R...we shall have *tangents* to S' corresponding to the *points* P, Q, R..., and these tangents will all pass through a point, viz. through the pole of L with respect to the auxiliary conic. Hence as many tangents to S' can be drawn through a point as there are points on S lying on a

straight line. That is to say the *class* [Art. 240] of S' is equal to the *degree* of S. Reciprocally the *degree* of S' is equal to the *class* of S.

In particular, if S be a conic it is of the second degree, and of the second class. Hence the reciprocal curve is of the second class, and of the second degree, and is therefore also a conic.

302. *To find the polar reciprocal of one conic with respect to another.*

Let the equation of the auxiliary conic be

$$\alpha x^2 + \beta y^2 + 1 = 0 \ldots\ldots\ldots\ldots\ldots \text{(i)};$$

and let the equation of the conic whose reciprocal is required be

$$ax^2 + by^2 + c + 2fy + 2gx + 2hxy = 0 \ldots \text{(ii)}.$$

The line $lx + my + n = 0$ will touch (ii) if

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0 \text{ (iii)}.$$

And, if the pole of $lx + my + n = 0$ with respect to (i) be (x', y'), its equation is the same as $\alpha x'x + \beta y'y + 1 = 0$.

Therefore $$\frac{l}{\alpha x'} = \frac{m}{\beta y'} = \frac{n}{1}.$$

Substitute, in (iii), and we have

$$A\alpha^2 x'^2 + B\beta^2 y'^2 + C + 2F\beta y' + 2G\alpha x' + 2H\alpha\beta x'y' = 0.$$

Hence the locus of the poles with respect to (i) of tangents to (ii) is the conic whose equation is

$$A\alpha^2 x^2 + B\beta^2 y^2 + C + 2F\beta y + 2G\alpha x + 2H\alpha\beta xy = 0.$$

303. The method of Reciprocal Polars enables us to obtain from any given theorem concerning the positions of points and lines, another theorem in which straight lines take the place of points and points of straight lines. Before proceeding to give examples of such reciprocal theorems we will give some simple cases of correspondence.

Points in one figure correspond to straight lines in the reciprocal figure.

The line joining two points in one figure corresponds to the point of intersection of the corresponding lines in the other.

The tangent to any curve in one figure corresponds to a point on the corresponding curve in the reciprocal figure.

The point of contact of a tangent corresponds to the tangent at the corresponding point.

If two curves touch, that is have two coincident points common, the reciprocal curves will have two coincident tangents common, and will therefore also touch.

The chord joining two points corresponds to the point of intersection of the corresponding tangents.

The chord of contact of two tangents corresponds to the point of intersection of tangents at the corresponding points.

Since the pole of any line through the centre of the auxiliary conic is at infinity, we see that the points at infinity on the reciprocal curve correspond to the tangents to the original curve from the centre of the auxiliary conic. Hence the reciprocal of a conic is an hyperbola, parabola, or ellipse, according as the tangents to it from the centre of the auxiliary conic are real, coincident, or imaginary; that is according as the centre of the auxiliary conic is outside, upon, or within the curve.

The following are examples of reciprocal theorems.

If the angular points of two triangles are on a conic, their six sides will touch another conic.	If the sides of two triangles touch a conic, their six angular points are on another conic.
The three intersections of opposite sides of a hexagon inscribed in a conic lie on a straight line. (*Pascal's Theorem*).	The three lines joining opposite angular points of a hexagon described about a conic meet in a point. (*Brianchon's Theorem*).
If the three sides of a triangle touch a conic, and two of its angular points lie on a second conic, the locus of the third angular point is a conic.	If the three angular points of a triangle lie on a conic, and two of its sides touch a second conic, the envelope of the third side is a conic.
If the sides of a triangle touch a conic, the three lines joining an angular point to the point of contact of the opposite side meet in a point.	If the angular points of a triangle lie on a conic, the three points of intersection of a side and the tangent at the opposite angular point lie on a line.

The polars of a given point with respect to a system of conics through four given points all pass through a fixed point.	The poles of a given straight line with respect to a system of conics touching four given straight lines all lie on a fixed straight line.
The locus of the pole of a given line with respect to a system of conics through four fixed points is a conic.	The envelope of the polar of a given point with respect to a system of conics touching four fixed lines is a conic.

304. We now proceed to consider the results which can be obtained by reciprocating with respect to a circle.

We know that the line joining the centre of a circle to any point P is perpendicular to the polar of P with respect to the circle. Hence, if P, Q be any two points, the angle between the polars of these points with respect to a circle is equal to the angle that PQ subtends at the centre of the circle. Reciprocally the angle between any two straight lines is equal to the angle which the line joining their poles with respect to a circle subtends at the centre of the circle.

We know also that the distances, from the centre of a circle, of any point and of its polar with respect to that circle, are inversely proportional to one another.

If we reciprocate with respect to a circle it is clear that a change in the radius of the auxiliary circle will make no change in the *shape* of the reciprocal curve, but only in its *size*. Hence, if we are not concerned with the absolute magnitudes of the lines in the reciprocal figure, we only require to know the *centre* of the auxiliary circle. We may therefore speak of reciprocating with respect to a *point* O, instead of with respect to a circle having O for centre.

305. If any conic be reciprocated with respect to a point O, the points on the reciprocal curve which correspond to the tangents through O to the original curve must be at an infinite distance.

The directions of the lines to the points at infinity on the reciprocal curve are perpendicular to the tangents from O to the original curve; and hence the angle between the asymptotes of the reciprocal curve is supplementary to the angle between the tangents from O to the original curve.

In particular, if the tangents from O to the original curve be at right angles, the reciprocal conic will be a rectangular hyperbola.

The *axes* of the reciprocal conic bisect the angles between its asymptotes. The axes are therefore parallel to the bisectors of the angles between the tangents from O to the original conic.

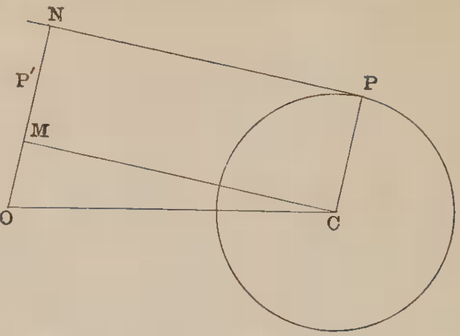
Corresponding to the points at infinity on the original conic we have the tangents to the reciprocal conic which pass through the origin. Hence the tangents from the origin to the reciprocal conic are perpendicular to the directions of the lines to the points at infinity on the original conic, so that the angle between the asymptotes of the original conic is supplementary to the angle between the tangents from the origin to the reciprocal conic.

In particular, if a rectangular hyperbola be reciprocated with respect to any point O, the tangents from O to the reciprocal conic will be at right angles to one another; in other words O is a point on the director-circle of the reciprocal conic.

306. The reciprocal of the origin is the line at infinity, and therefore the reciprocal of the polar of the origin is the pole of the line at infinity. That is to say, the *polar of the origin* reciprocates into the *centre* of the reciprocal conic.

307. As an example of reciprocation take the known theorem—"If two of the conics which pass through four given points are rectangular hyperbolas, they will all be rectangular hyperbolas." If this be reciprocated with respect to any point O we obtain the following, "If the director-circles of two of the conics which touch four given straight lines pass through a point O, the director-circles of all the conics will pass through O." Whence we have "The director-circles of all conics which touch four given straight lines have a given radical axis."

308. *To find the polar reciprocal of one circle with respect to another.*



Let C be the centre and a be the radius of the circle to be reciprocated, O the centre and k the radius of the auxiliary circle, and let c be the distance between the centres of the two circles.

Let PN be any tangent to the circle C, and let P' be its pole with respect to the auxiliary circle. Let OP' meet the tangent in the point N, and draw CM perpendicular to ON.

Then $$OP' \,.\, ON = k^2;$$

$$\therefore \quad \frac{k^2}{OP'} = ON = OM + MN = c \cos COM + a.$$

Hence the equation of the locus of P' is

$$\frac{\frac{k^2}{a}}{r} = 1 + \frac{c}{a} \cos\theta.$$

This is the equation of a conic having O for *focus*, $\frac{k^2}{a}$ for *semi-latus rectum*, and $\frac{c}{a}$ for *eccentricity*. The directrix of the conic is the line whose equation is

$$\frac{k^2}{r} = c \cos\theta, \quad \text{or } x = \frac{k^2}{c}.$$

Hence the directrix of the reciprocal curve is the polar of the centre of the original circle.

It is clear from the value found above for the eccentricity, that the reciprocal curve is an *ellipse* if the point O be *within* the circle C, an hyperbola if O be *outside* that circle, and a *parabola* if O be *upon* the circumference of the circle.

Ex. 1. *Tangents to a conic subtend equal angles at a focus.*

Reciprocate with respect to the focus:—then corresponding to the two tangents to the conic, there are two points on a circle; the point of intersection of the tangents to the conic corresponds to the line joining the two points on the circle; and the points of contact of the tangents to the conic correspond to the tangents at the points on the circle. Also the angle subtended at the focus of the conic by any two points is equal to the angle between the lines corresponding to those two points. Hence the reciprocal theorem is—The line joining two points on a circle makes equal angles with the tangents at those points.

Ex. 2. *The envelope of the chord of a conic which subtends a right angle at a fixed point O is a conic having O for a focus, and the polar of O, with respect to the original conic, for the corresponding directrix.*

Reciprocate with respect to O, and the proposition becomes—The locus of the point of intersection of tangents to a conic which are at right angles to one another is a concentric circle.

Ex. 3. *If two conics have a common focus, two of their common chords will pass through the intersection of their directrices.*

Reciprocate with respect to the common focus, and the proposition becomes—Two of the points of intersection of the common tangents to two circles are on the line joining the centres of the circles.

Ex. 4. *The orthocentre of a triangle circumscribing a parabola is on the directrix.*

Reciprocating with respect to the orthocentre we obtain—A conic circumscribing a triangle and passing through the orthocentre is a rectangular hyperbola.

Many of the examples on Chapter VIII. are easily proved by reciprocation: for example, the reciprocal of 23 with respect to the common focus is—circles are described with equal radii, and with their centres on a second circle; prove that they all touch two fixed circles, whose radii are the sum and difference respectively of the radii of the moving circle and of the second circle, and which are concentric with the second circle.

309. If we have a system of circles with the same radical axis we can reciprocate them into a system of confocal conics.

If we reciprocate with respect to *any* point O we obtain a system of conics having O for one focus, and [Art. 306] the centre of any conic is the reciprocal of the polar of O with respect to the corresponding circle. Now either of the two 'limiting points' of the system is such that its polar with respect to any circle of the system is a fixed straight line, namely a line through the other limiting point parallel to the radical axis. If therefore the system of circles be reciprocated with respect to a limiting point the reciprocals will have the same centre; and if they have a common centre and one common focus they will be confocal. Since the radical axis is parallel to and midway between a limiting point and its polar, the reciprocal of the radical axis (with respect to the limiting point) is on the line through the focus and centre of the reciprocal conics, and is twice as far from the focus as the centre; so that when we reciprocate a system of coaxial circles with respect to a limiting point, the radical axis reciprocates into the other focus of the system of confocal conics.

The following theorems are reciprocal:

The tangents at a common point of two confocal conics are at right angles.	The points of contact of a common tangent to two circles subtend a right angle at one of the limiting points.
The locus of the point of intersection of two lines, each of which touches one of two confocal conics, and which are at right angles to one another, is a circle.	The envelope of the line joining two points, each of which is on one of two circles, and which subtend a right angle at a limiting point, is a conic one of whose foci is at the limiting point.
If from any point two pairs of tangents P, P' and Q, Q' be drawn to two confocal conics; the angle between P and Q is equal to that between P' and Q'.	If any straight line cut two circles in the points P, P' and Q, Q'; the angles subtended at a limiting point by PQ and $P'Q'$ are equal.

From any point four tangents P, P' and Q, Q' are drawn to two confocal conics, and the point of contact of P is joined to the points of contact of Q, Q'; shew that these lines make equal angles with the tangent P. [Art. 229.]

Any line cuts two circles in P, P' and Q, Q' respectively; and the tangent at P cuts the tangents at Q, Q' in q, q'; shew that Pq, Pq' subtend equal (or supplementary) angles at a limiting point.

PROJECTION.

310. If any point P be joined to a fixed point V, and VP be cut by any fixed plane in P', the point P' is called the projection of P on that plane. The point V is called the *vertex* or the *centre of projection*, and the cutting plane is called the *plane of projection*.

311. *The projection of any straight line is a straight line.*

For the straight lines joining V to all the points of any straight line are in a plane, and this is cut by the plane of projection in a straight line.

312. *Any plane curve is projected into a curve of the same degree.*

For, if any straight line meet the original curve in any number of points A, B, C, D..., the projection of the line will meet the projection of the curve where VA, VB, VC, VD... meet the plane of projection. There will therefore be the *same number* of points on a straight line in the one curve as in the other. This proves the proposition.

In particular, *the projection of a conic is a conic.*

This proposition includes the geometrical theorem that every plane section of a right circular cone is a conic.

313. *A tangent to a curve projects into a tangent to the projected curve.*

For, if a straight line meet a curve in two points A, B, the projection of that line will meet the projected curve in two points a, b where VA, VB meet the plane of projection. Now if A and B coincide, so also will a and b.

314. *The relation of pole and polar with respect to a conic are unaltered by projection.*

This follows from the two preceding Articles.

It is also clear that two conjugate points, or two conjugate lines, with respect to a conic, project into conjugate points, or lines, with respect to the projected conic.

315. Draw through the vertex a plane parallel to the plane of projection, and let it cut the original plane in the line $K'L'$. Then, since the plane $VK'L'$ and the plane of projection are parallel, their line of intersection, which is the projection of $K'L'$, is at an infinite distance.

Hence to project any particular straight line $K'L'$ to an infinite distance, take any point V for vertex and a plane parallel to the plane $VK'L'$ for the plane of projection.

Straight lines which meet in any point on the line $K'L'$ will be projected into parallel straight lines, for their point of intersection will be projected to infinity.

316. A system of parallel lines on the original plane will be projected into lines which meet in a point.

For, let VP be the line through the vertex parallel to the system, P being on the plane of projection; then, since VP is in the plane through V and any one of the parallel lines, the projection of every one of the parallel lines will pass through P.

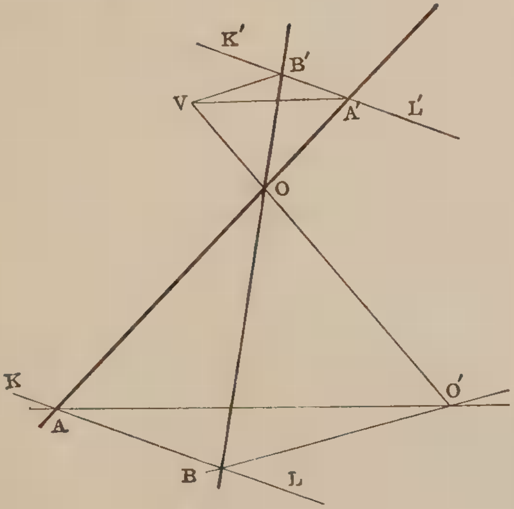
For different systems of parallel lines the point P will change; but, since VP is always parallel to the original plane, the point P is always on the line of intersection of the plane of projection and a plane through the vertex parallel to the original plane.

Hence any system of parallel lines on the original plane is projected into a system of lines passing through a point, and all such points, for different systems of parallel lines, are on a straight line.

317. Let KL be the line of intersection of the original plane and the plane of projection. Draw through the vertex a plane parallel to the plane of projection, and let

it cut the original plane in the line $K'L'$. Let the two straight lines AOA', BOB' meet the lines KL, $K'L'$ in the points A, B and A', B' respectively; and let VO meet the plane of projection in O'. Then AO' and BO' are the projections of AOA' and BOB'.

Since the planes $VA'B'$, $AO'B$ are parallel, and parallel planes are cut by the same plane in parallel lines, the lines VA', VB' are parallel respectively to AO', BO'. The angle $A'VB'$ is therefore equal to the angle $AO'B$, that is, $A'VB'$ is equal to the angle into which AOB is projected.



Similarly, if the straight lines CD, ED, meet $K'L'$ in C', D' respectively, the angle $C'VD'$ will be equal to the angle into which CDE is projected.

From the above we obtain the fundamental proposition in the theory of projections, viz.,

Any straight line can be projected to infinity, and at the same time any two angles into given angles.

For, let the straight lines bounding the two angles meet the line which is to be projected to infinity in the points

A', B' and C', D'; draw any plane through $A'B'C'D'$, and in that plane draw segments of circles through A', B' and C', D' respectively containing angles equal to the two given angles. Either of the points of intersection of these segments of circles may be taken for the centre of projection, and the plane of projection must be taken parallel to the plane we have drawn through $A'B'C'D'$.

If the segments do not meet, the centre of projection is imaginary.

Ex. 1. *To shew that any quadrilateral can be projected into a square.*

Let $ABCD$ be the quadrilateral; and let P, Q [see figure to Art. 60] be the points of intersection of a pair of opposite sides, and let the diagonals BD, AC meet the line PQ in the points S, R. Then, if we project PQ to infinity and at the same time the angles PDQ and ROS into right angles, the projection must be a square. For, since PQ is projected to infinity, the pairs of opposite sides of the projection will be parallel, that is to say, the projection is a parallelogram; also one of the angles of the parallelogram is a right angle, and the angle between the diagonals is a right angle; hence the projection is a square.

Ex. 2. *To shew that the triangle formed by the diagonals of a quadrilateral is self-polar with respect to any conic which touches the sides of the quadrilateral.*

Project the quadrilateral into a square; then, the circle circumscribing the square is the director-circle of the conic, therefore the intersection of the diagonals of the square is the centre of the conic.

Now the polar of the centre is the line at infinity; hence the polar of the point of intersection of two of the diagonals is the third diagonal.

Ex. 3. *If a conic be inscribed in a quadrilateral the line joining two of the points of contact will pass through one of the angular points of the triangle formed by the diagonals of the quadrilateral.*

Ex. 4. *If ABC be a triangle circumscribing a parabola, and the parallelograms $ABA'C$, $BCB'A$, and $CAC'B$ be completed; then the chords of contact will pass respectively through A', B', C'.*

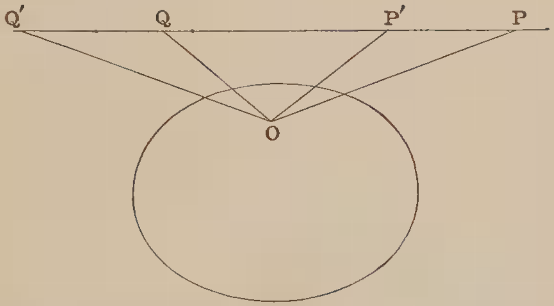
This is a particular case of Ex. 3, one side of the quadrilateral being the line at infinity.

Ex. 5. *If the three lines joining the angular points of two triangles meet in a point, the three points of intersection of corresponding sides will lie on a straight line.*

Project two of the points of intersection of corresponding sides to infinity, then two pairs of corresponding sides will be parallel, and it is easy to shew that the third pair will also be parallel.

Ex. 6. *Any two conics can be projected into concentric conics.* [See Art. 283.]

318. *Any conic can be projected into a circle having the projection of any given point for centre.*



Let O be the point whose projection is to be the centre of the projected curve.

Let P be any point on the polar of O, and let OQ be the polar of P; then OP and OQ are conjugate lines.

Take OP', OQ' another pair of conjugate lines.

Then project the polar of O to infinity, and the angles POQ, $P'OQ'$ into right angles. We shall then have a conic whose centre is the projection of O, and since two pairs of conjugate diameters are at right angles, the conic is a circle.

319. *A system of conics inscribed in a quadrilateral can be projected into confocal conics.*

Let two of the sides of the quadrilateral intersect in the point A, and the other two in the point B. Draw any conic through the points A, B, and project this conic into a circle, the line AB being projected to infinity; then, A, B are projected into the circular points at infinity, and since the tangents from the circular points at infinity to all the conics of the system are the same, the conics must be confocal.

Ex. 1. *Conics through four given points can be projected into coaxial circles.*

For, project the line joining two of the points to infinity, and one of the conics into a circle; then all the conics will be projected into circles, for they all go through the circular points at infinity.

Ex. 2. *Conics which have double contact with one another can be projected into concentric circles.*

Ex. 3. *The three points of intersection of opposite sides of a hexagon inscribed in a conic lie on a straight line.* [Pascal's Theorem.]

Project the conic into a circle, and the line joining the points of intersection of two pairs of opposite sides to infinity; then we have to prove that if two pairs of opposite sides of a hexagon inscribed in a circle are parallel, the third pair are also parallel.

Ex. 4. *Shew that all conics through four fixed points can be projected into rectangular hyperbolas.*

There are three pairs of lines through the four points, and if two of the angles between these pairs of lines be projected into right angles, all the conics will be projected into rectangular hyperbolas. [Art. 187, Ex. 1.]

Ex. 5. *Any three chords of a conic can be projected into equal chords of a circle.*

Let AA', BB', CC' be the chords; let AB', $A'B$ meet in K, and AC', $A'C$ in L. Project the conic into a circle, KL being projected to infinity.

Ex. 6. *If two triangles are self polar with respect to a conic, their six angular points are on a conic, and their six sides touch a conic.*

Let the triangles be ABC, $A'B'C'$. Project BC to infinity, and the conic into a circle; then A is projected into the centre of the circle, and AB, AC are at right angles, since ABC is self polar; also, since $A'B'C'$ is self polar with respect to the circle, A is the orthocentre of the triangle $A'B'C'$.

Now a rectangular hyperbola through A', B', C' will pass through A, and a rectangular hyperbola through B will go through C. Hence, since a rectangular hyperbola can be drawn through any four points, the six points A, B, C, A', B', C' are on a conic.

Also a parabola can be drawn to touch the four straight lines $B'C'$, $C'A'$, $A'B'$, AB. And A is on the directrix of the parabola [Art. 107 (3)]; therefore AC is a tangent. Hence a conic touches the six sides of the two triangles.

320. Properties of a figure which are true for any projection of that figure are called *projective properties*. In general such properties do not involve magnitudes. There are however some projective properties in which the magnitudes of lines and angles are involved: the most important of these is the following:—

The cross ratios of pencils and ranges are unaltered by projection.

Let A, B, C, D be four points in a straight line, and A', B', C', D' be their projections. Then, if V be the centre of projection, VAA', VBB', VCC', VDD' are straight lines; and we have [Art. 55]

$$\{ABCD\} = V\{ABCD\} = \{A'B'C'D'\}.$$

If we have any pencil of four straight lines meeting in O, and these be cut by any transversal in A, B, C, D; then

$$O\{ABCD\} = \{ABCD\} = V\{ABCD\} = \{A'B'C'D'\} \\ = O'\{A'B'C'D'\}.$$

From the above together with Article 62 it follows that *if any number of points be in involution, their projections will be in involution.*

Ex. 1. *Any chord of a conic through a given point O is divided harmonically by the curve and the polar of O.*

Project the polar of O to infinity, then O is the centre of the projection, the chord therefore is bisected in O, and $\{POQ\infty\}$ is harmonic when $PO = OQ$.

Ex. 2. *Conics through four fixed points are cut by any straight line in pairs of points in involution.* [Desargue's Theorem].

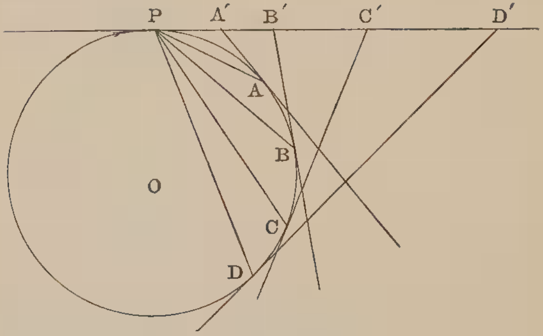
Project two of the points into the circular points at infinity, then the conics are projected into co-axial circles, and the proposition is obvious.

321. *The cross ratio of the pencil formed by four intersecting straight lines is equal to that of the range formed by their poles with respect to any conic.*

Since the cross ratios of pencils and ranges are unaltered by projection, we may project the conic into a circle. Now in a circle any straight line is perpendicular to the line joining the centre of the circle to its pole with respect to the circle. Hence the cross ratio of the pencil formed by four intersecting straight lines is equal to that of the pencil subtended at the centre of the circle by their poles, and therefore equal to the cross ratio of the range formed by their poles.

322. *The cross ratio of the pencil formed by joining any point on a conic to four fixed points is constant, and is equal to that of the range in which the tangents at those points are cut by any tangent.*

Since the cross ratios of pencils and ranges are unaltered by projection, we need only prove the proposition for a *circle*.



Let A, B, C, D be four fixed points on a circle; let P be any other point on the circle, and let the tangent at P meet the tangents at A, B, C, D in the points A', B', C', D'.

Then, if O be the centre of the circle, OA' is perpendicular to PA, OB' to PB, OC' to PC, and OD' to PD.

Hence

$$\{A'B'C'D'\} = O\{A'B'C'D'\} = P\{ABCD\}.$$

But the angles APB, BPC, CPD are constant, since A, B, C, D are fixed points.

Therefore $\{A'B'C'D'\} = P\{ABCD\} = \text{const.}$

If Q be any point which is not on the circle, $Q\{ABCD\}$ cannot be equal to $P\{ABCD\}$; this is seen at once if we take P such that APQ is a straight line, and consider the ranges made on BC by the two pencils. Hence we have the following converse proposition.

If a point P *move so that the cross ratio of the pencil formed by joining it to four fixed points* A, B, C, D, *is constant;* P *will describe a conic passing through* A, B, C, D.

Ex. 1. *The four extremities of two conjugate chords of a conic subtend a harmonic pencil at any point on the curve.*

Let the chords be AC, BD; let E be the pole of BD, and let F be the point of intersection of AC, BD. The four points subtend, at all points on the curve, pencils of equal cross ratio. Take a point indefinitely near to D; then the pencil is $D\{ABCE\}$. But the range A, B, C, E is harmonic, which proves the proposition.

Ex. 2. *If two triangles circumscribe a conic, their six angular points are on another conic.*

Let ABC, $A'B'C'$ be the two triangles. Let $B'C'$ cut AB, AC in E', D', and let BC cut $A'B'$, $A'C'$ in E, D. Then the ranges made on the four tangents AB, AC, $A'B'$, $A'C'$ by the two tangents BC, $B'C'$ are equal.

Hence
$$\{BCED\}=\{E'D'B'C'\};$$
$$\therefore \quad A'\{BCED\}=A\{E'D'B'C'\},$$
$$\text{or} \quad A'\{BCB'C'\}=A\{BCB'C'\},$$
which proves the proposition.

The proposition may also be proved by projecting B, C into the circular points at infinity; the conic is thus projected into a parabola, of which A is the focus; and it is known that the *circle* circumscribing $A'B'C'$ will pass through A.

323. Def. Ranges and pencils are said to be *homographic* when every four constituents of the one, and the corresponding four constituents of the other, have equal cross ratios.

Another definition of homographic ranges or pencils is the following:—two ranges or pencils are said to be homographic which are so connected that to each point or line of the one system corresponds one, and only one, point of the other.

To show that this definition of homographic ranges is equivalent to the former, let the distances, measured from fixed points, of any two corresponding points of the two systems be x, y; then we must have an equation of the form

$$x = \frac{ay+b}{cy+d}.$$

The proposition follows from the fact that the cross ratio of every four points of the one system, namely

$$\frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_4)(x_3-x_2)},$$

is not altered if we substitute $\dfrac{ay_1+b}{cy_1+d}$ for x_1, and similar expressions for x_2, x_3 and x_4.

Ex. 1. *The points of intersection of corresponding lines of two homographic pencils describe a conic.*

Let P, Q, R, S be four of the points of intersection, and O, O' the vertices of the pencils.

Then $O\{PQRS\}=O'\{PQRS\}$; therefore [Art. 322] O, O', P, Q, R, S are on a conic. But five points are sufficient to determine a conic; hence the conic through O, O' and any three of the intersections will pass through every other intersection.

Ex. 2. *The lines joining corresponding points of two homographic ranges envelope a conic.*

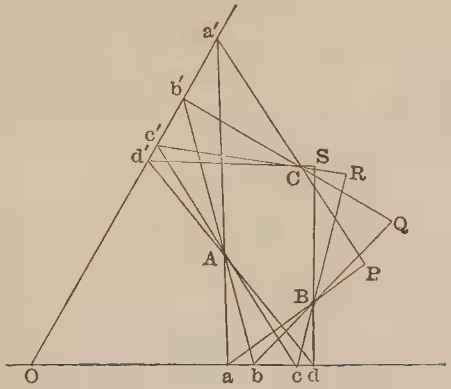
Let a, b, c, d be any four of the points of one system, and a', b', c', d' be the corresponding points of the other system. Then aa', bb', cc', dd' are cut by the fixed lines in ranges of equal cross ratio. Hence a conic will touch the fixed lines, and also aa', bb', cc', dd'. But five tangents are sufficient to determine a conic; hence the conic which touches the fixed lines, and three of the lines joining corresponding points of the ranges, will touch all the others.

Ex. 3. *Two angles PAQ, PBQ of constant magnitude move about fixed points A, B, and the point P describes a straight line; shew that Q describes a conic through A, B.* [Newton.]

Corresponding to one position of AQ, there is one, and only one, position of BQ. Hence, from Ex. 1, the locus of Q is a conic.

Ex. 4. *The three sides of a triangle pass through fixed points, and the extremities of its base lie on two fixed straight lines; shew that its vertex describes a conic.* [Maclaurin.]

Let A, B, C be the three fixed points, and let Oa, Oa' be the two fixed straight lines. Suppose triangles drawn as in the figure.



Then the ranges $\{abcd...\}$ and $\{a'b'c'd'...\}$ are homographic. Therefore the pencils $B\,\{abcd...\}$ and $C\,\{a'b'c'd'...\}$ are homographic.

Ex. 5. *If all the sides of a polygon pass through fixed points, and all the angular points but one move on fixed straight lines; the remaining angular point will describe a conic.*

Ex. 6. *A, A' are fixed points on a conic, and from A and A' pairs of tangents are drawn to any confocal conic, which meet the original conic in C, D and C', D'; shew that the locus of the point of intersection of CD and $C'D'$ is a conic.*

The tangents from A to a confocal are equally inclined to the tangent at A [Art. 228, Cor. 3], therefore the chord CD cuts the tangent at A in some fixed point O [Art. 195, Ex. 2]. So also $C'D'$ passes through a fixed point O'. Now if we draw any line OCD through O, one confocal, and only one, will touch the lines AC, AD; and the tangents from A' to this confocal will determine C' and D', so that corresponding to any position of OCD there is one, and only one, position of $O'C'D'$. The locus of the intersection is therefore a conic from Ex. 1.

Ex. 7. *If AOA', BOB', COC', DOD'... be chords of a conic, and P any point on the curve, then will the pencils $P\{ABCD...\}$ and $P\{A'B'C'D'...\}$ be homographic.*

Project the conic into a circle having O for centre.

Ex. 8. *If there are two systems of points on a conic which subtend homographic pencils at any point on the curve, the lines joining corresponding points of the two systems will envelope a conic having double contact with the original conic.*

Let $A, B, C, D\ldots$, and $A', B', C', D'\ldots$ be the two systems of points. Project AA', BB', CC' into equal chords of a circle [Art. 319, Ex. 5]; let P, P' be *any* pair of corresponding points, and O any point on the circle; then we have $O\{ABCP\} = O\{A'B'C'P'\}$. Hence PP' is equal to AA', and therefore the envelope of PP' is a concentric circle.

Ex. 9. *If a polygon be inscribed in a conic, and all its sides but one pass through fixed points, the envelope of that side will be a conic.*

This follows from Ex. 7 and Ex. 8.

324. *Any two lines at right angles to one another, and the lines through their intersection and the circular points at infinity, form a harmonic pencil.*

Let the two lines at right angles to one another be $xy = 0$, then the lines to the circular points at infinity will be given by $x^2 + y^2 = 0$. By Art. 58 these two pairs of lines are harmonically conjugate.

We may also shew that two lines which are inclined at any constant angle, and the lines to the circular points at infinity, form a pencil of constant cross ratio.

Ex. *The locus of the point of intersection of two tangents to a conic which divide a given line AB harmonically is a conic through A, B, and the envelope of the chord of contact is a conic which touches the tangents to the original conic from A, B.*

Project A, B into the circular points at infinity and the proposition becomes: the locus of the point of intersection of two tangents to a conic which are at right angles to one another is a circle; and the envelope of the chord of contact is a confocal conic.

325. The following are additional examples of the methods of reciprocation and projection.

Ex. 1. *If the sides of a triangle touch a conic, and if two of the angular points move on fixed confocal conics, the third angular point will describe a confocal conic.*

Let ABC, $A'B'C'$ be two indefinitely near positions of the triangle, and let AA', BB', CC' produced form the triangle PQR. The six points A, B, C, A', B', C' are on a conic [Art. 322, Ex. 2], and this conic will ultimately touch the sides of PQR in the points A, B, C. Hence PA, QB, RC will meet in a point [Art. 186, Ex. 3]; and it is easily seen that the pencils $A\{QCPB\}$, $B\{RAQC\}$, $C\{PBRA\}$ are harmonic. Now, if A move on a conic confocal to that which AB, AC touch, the tangent at A, that is the line QR, will make equal angles with AB, AC. Hence, since $A\{QCPB\}$ is harmonic, PA is perpendicular to QR. Similarly, if B move on a confocal, QB is perpendicular to RP. Hence RC must be perpendicular to PQ, and therefore CA, CB make equal angles with PQ; whence it follows that C moves on a confocal conic.

[The proposition can easily be extended. For, let $ABCD$ be a quadrilateral circumscribing a conic, and let A, B, C move on confocals. Let DA, CB meet in E, and AB, DC in F. Then, by considering the triangles ABE, BCF, we see that E and F move on confocals. Hence, by considering the triangle CED, we see that D will move on a confocal.]

If we reciprocate with respect to a focus we obtain the following theorem:

If the angular points of a triangle are on a circle of a co-axial system, and two of the sides touch circles of the system, the third side will touch another circle of the system. [Poncelet's theorem.]

Ex. 2. *The six lines joining the angular points of a triangle to the points where the opposite sides are cut by a conic, will touch another conic.*

The reciprocal theorem is:—

The six points of intersection of the sides of a triangle with the tangents to a conic drawn from the opposite angular points, will lie on another conic.

Project two of the points into the circular points at infinity, then the opposite angular point of the triangle will be projected into a focus, and we have the obvious theorem:—

Two lines through a focus of a conic are cut by pairs of tangents parallel to them in four points on a circle.

Ex. 3. The following theorems are deducible from one another.

(i) *Two lines at right angles to one another are tangents one to each of two confocal conics; shew that the locus of their intersection is a circle, and that the envelope of the line joining their points of contact is another confocal.*

(ii) *Two points, one on each of two co-axial circles, subtend a right angle at a limiting point; shew that the envelope of the line joining them is a conic with one focus at the limiting point, and that the locus of the intersection of the tangents at the points is a co-axial circle.*

(iii) *Two lines which are tangents one to each of two conics, cut a diagonal of their circumscribing quadrilateral harmonically; shew that the locus of the intersection of the lines is a conic through the extremities of that diagonal, and that the envelope of the line joining the points of contact is a conic inscribed in the same quadrilateral.*

(iv) *AOB, COD are common chords of two conics, and P, Q are points, one on each conic, such that $O\{APBQ\}$ is harmonic; shew that the envelope of the line PQ is a conic touching AB, CD, and that the tangents at P, Q meet on a conic through A, B, C, D.*

(v) *If two points be taken, one on each of two circles, equidistant from their radical axis, the envelope of the line joining them is a parabola which touches the radical axis, and the locus of the intersection of the tangents at the points is a circle through their common points.*

Examples on Chapter XIV.

1. Shew that an hyperbola is its own reciprocal with respect to the conjugate hyperbola.

2. Shew that a system of conics through four fixed points can be reciprocated into concentric conics.

3. Shew that four conics can be described having a common focus and passing through three given points, and that the latus rectum of one of these is equal to the sum of the latera recta of the other three. Shew also that their directrices meet two and two on the sides of the triangle.

4. If each of two conics be reciprocated with respect to the other; shew that the two conics and the two reciprocals have a common self-conjugate triangle.

5. Two conics L_1 and L_2 are reciprocals with respect to a conic U. If M_1 be the reciprocal of L_1 with respect to L_2, and M_2 be the reciprocal of L_2 with respect to L_1; shew that M_1 and M_2 are reciprocals with respect to U.

6. If two pairs of conjugate rays of a pencil in involution be at right angles, every pair will be at right angles.

7. If two pairs of points in an involution have the same point of bisection, every pair will have the same point of bisection. Where is the centre of the involution?

8. The pairs of tangents from any point to a system of conics which touch four fixed straight lines form a pencil in involution. Hence shew that the director circles of the system have a common radical axis.

9. Two circles and their centres of similitude subtend a pencil in involution at any point.

10. If two finite lines be divided into the same number of parts, the lines joining corresponding points will envelope a parabola.

11. If P, P' be corresponding points of two homographic ranges on the lines OA, OA', and the parallelogram $POP'Q$ be completed; shew that the locus of Q is a conic.

12. Three conics have two points common; shew that the three lines joining their other intersections two and two meet in a point, and that any line through that point is cut by the conics in six points in involution.

13. Shew that, if the three points of intersection of corresponding sides of two triangles lie on a straight line, the two triangles can both be projected into equilateral triangles.

14. Shew that any three angles may be projected into right angles.

15. A, B, C are three fixed points on a conic; find geometrically a point on the curve at which AB, BC subtend equal angles.

16. Through a fixed point O any line is drawn cutting the sides of a given triangle in A', B', C' respectively, and P is the point on the line such that $\{A'B'C'P\}$ is harmonic; shew that the locus of P is a conic.

17. When four conics pass through four given points, the pencil, formed by the polars of any point with respect to them, is of constant cross ratio.

18. If two angles, each of constant magnitude, turn about their vertices, in such a manner that the point of intersection of two of their sides is on a conic through the vertices, the other two sides will intersect on a second conic through their vertices.

19. If all the angular points of a polygon move on fixed straight lines, and all the sides but one turn about fixed points, the free side of the polygon will envelope a conic.

20. If a polygon be circumscribed to a conic, and all its angular points but one lie on fixed straight lines, the locus of that angular point will be a conic.

APPENDIX.

ANSWERS &c. TO THE EXAMPLES.

CHAPTER II.

3. Ans. $p_0(x-a)^n - p_1(x-a)^{n-1}(y-b) + p_2(x-a)^{n-2}(y-b)^2 - \dots - + (-1)^n p_n (y-b)^n = 0$. 4. The lines make equal angles with one another. 5. Take OA, OB for axes, and let OA, OB, OP, OQ be a, b, h, k respectively. Since $AP = c\,.\,BQ$, we have $h - a = c(k-b)$. If (x, y) be middle point of PQ, $2x = h$, $2y = k$; whence required locus is $2x - a = c(2y-b)$. 7. Take the fixed lines for axes and let P be (x, y) and Q be (x', y'). Then $x' = x + y\cos\omega$, $y' = y + x\cos\omega$. Find x and y in terms of x' and y', and substitute in the equation of the locus of P. 8. Use polar co-ordinates with O for pole. 10. The equations of AB, AD, BC, CD are $\theta = 0$, $\theta = \alpha$, $r\sin(\theta - \alpha) + a\sin\alpha = 0$, and $r\sin\theta = b\sin\alpha$; where a, b are the lengths of AB and AD, and α is the angle BAD. The equation of AC is $\theta = \tan^{-1}\dfrac{b\sin\alpha}{a + b\cos\alpha}$, and of BD is $ra\sin\theta - ab\sin\alpha + br\sin(\alpha - \theta) = 0$. 14. Ans. $7y - 3x - 19 = 0$, $7x + 3y - 33 = 0$, $7y - 3x + 10 = 0$, and $7x + 3y - 4 = 0$. 15. If the base be taken as axis of x, the *sum* of the positive angles the sides make with it is constant. 16. The equation of the locus is $\dfrac{y^2 + (x-a)(x-b)}{a-b} = \dfrac{y^2 + (x-c)(x-d)}{c-d}$. The points are on the axis of x and a, b, c, d are their distances from the origin. 18. The equation of the locus is $(a-a')xy + l(x-a)(x-a') = 0$, where AB is axis of x, the other given line the axis of y, and $OA = a$, $OB = a'$, and l the intercept on axis of y. 21. The bisectors of the angles between the two pairs of straight lines coincide [Art. 39]. 22. Ans. $(ab' - a'b)^2 = 4(ha' - h'a)(h'b - hb')$. 23. This is reduced to the preceding by means of question 2. 26. The result easily follows from the polar form of the equation, viz. $m\tan 3\theta + 1 = 0$. 29. If the straight lines be $y = m_1x$, $y = m_2x$, and $y = m_3x$; we have $k^6 = \dfrac{(y - m_1x)^2(y - m_2x)^2(y - m_3x)^2}{(1 + m_1^2)(1 + m_2^2)(1 + m_3^2)}$. But

$ay^3 + by^2x + cyx^2 + dx^3 \equiv a(y - m_1x)(y - m_2x)(y - m_3x)$, and $(1 + m_1^2)(1 + m_2^2)(1 + m_3^2) = 1 + (m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1) + (m_1m_2 + m_2m_3 + m_3m_1)^2 - 2m_1m_2m_3(m_1 + m_2 + m_3) + m_1^2m_2^2m_3^2 = 1 + \frac{b^2}{a^2} - 2\frac{c}{a} + \frac{c^2}{a^2} - 2\frac{db}{a^2} + \frac{d^2}{a^2}$; whence the result. 30. The equation of any pair of perpendicular lines is $x^2 + \lambda xy - y^2 = 0$. Hence given equation must be equivalent to $(Ex + Fy)(x^2 + \lambda xy - y^2) = 0$, so that $E = A$, $F = -D$, $F + \lambda E = 3B$, and $\lambda F - E = 3C$. 33. The lines are $ax^2 + 2hxy + by^2 - \frac{g}{g'}(a'x^2 + 2h'xy + b'y^2) = 0$ [Art. 38]. 34. Let A, B, C be (a_1, b_1), (a_2, b_2) and (a_3, b_3); and A', B', C' be (α_1, β_1), (α_2, β_2), (α_3, β_3). The equations of the three perpendiculars from A', B', C' on the sides of ABC are $x(a_2 - a_3) + y(b_2 - b_3) - \alpha_1(a_2 - a_3) - \beta_1(b_2 - b_3) = 0$ (1), $x(a_3 - a_1) + y(b_3 - b_1) - \alpha_2(a_3 - a_1) - \beta_2(b_3 - b_1) = 0$ (2), and $x(a_1 - a_2) + y(b_1 - b_2) - \alpha_3(a_1 - a_2) - \beta_3(b_1 - b_2) = 0$ (3). If (1), (2), (3) meet in a point the sum of the constants is zero, and this sum can be written in the symmetrical form $a_1\alpha_2 - a_2\alpha_1 + a_2\alpha_3 - a_3\alpha_2 + a_3\alpha_1 - a_1\alpha_3 + b_1\beta_2 - b_2\beta_1 + b_2\beta_3 - b_3\beta_2 + b_3\beta_1 - b_1\beta_3 = 0$.

CHAPTER IV.

4. The locus is $(1 - n^2)(x^2 + y^2 + a^2) - 2a(1 + n^2)x = 0$, where $(a, 0)$, $(-a, 0)$ are the two points A, B. The common radical axis is $x = 0$. 6. Ans. $x^2 + y^2 + 2dx + 2ey + f(Ax + By) = 0$. 7. Ans. $2x^2 + 2y^2 + 2x + 6y + 1 = 0$. 8. See Art. 38. 9 and 10. Substitute $\frac{k^2}{r}$ for r in the polar equation of the line or of the circle. 12. If a common tangent, PQ, of two of the circles cut the radical axis in O, the tangents from O to all the other circles of the system, *including the limiting circles*, will be equal to OP; therefore the limiting points are on a circle on PQ as diameter. 13. If one circle is within the other, (1) the radical axis must cut in imaginary points, therefore b is positive; (2) the centres must be on the same side of the radical axis, therefore a and a' have the same sign. 15. See Art. 86. 19. We may take

$$x\cos\alpha + y\sin\alpha - a = 0,\ x\cos\left(\alpha + \frac{2\pi}{n}\right) + y\sin\left(\alpha + \frac{2\pi}{n}\right) - a = 0, \ \&c.$$

for the equations of the sides. The sum of the squares of the perpendiculars from (x, y) is sum of squares of left sides; and in this sum the coefficients of x^2 and y^2 are equal, since $\cos 2\alpha + \cos 2\left(\alpha + \frac{2\pi}{n}\right) + \ldots = 0$; also the coefficient of xy is zero, since $\sin 2\alpha + \sin 2\left(\alpha + \frac{2\pi}{n}\right) + \ldots = 0$.

20. $\frac{h}{x}+\frac{k}{y}=2$, (h, k) being the point through which PQ passes. 21. If P be any point on the circle, and A, B the ends of a diameter, $PA^2+PB^2=AB^2$; express this in polar co-ordinates. 22. Eliminate θ. Condition for tangency is $p=2a\cos^2\frac{\beta}{2}$ or $p=-2a\sin^2\frac{\beta}{2}$. 23. $x=\frac{5}{8}$, $y=-\frac{5}{4}$. 24. Two circles. 25. The whole lengths of the lines, from the points of contact to their intersection, are equal to one another. 26. The given lines must intersect on the radical axis of the circles. 27. If given points are $(\pm a, 0)$, and tangents are parallel to $y=x\tan\theta$, the equation of the locus is $y^2+2xy\cot\theta-x^2+a^2=0$. 29. For straight lines, $A+B+C=0$. 31. Any circle through $(\pm a, 0)$ is $x^2+y^2-2by-a^2=0$. The orthogonal circles are $x^2+y^2-2cx+a^2=0$. 33. Take $x^2+y^2-2ax=0$, $x^2+y^2-2by=0$ for the equations of the circles. 35. The equation of the locus is $(b^2+c^2)(x^2+y^2+a^2)+2a(b^2-c^2)x\pm 4abcy=(x^2+y^2-a^2)^2$. The bisectors touch the circles $x^2+(y\pm a)^2=\frac{1}{2}(b\pm c)^2$. 36. The centre of the required circle must be the radical centre of the three escribed circles. The equation of circle touching BC, and AB, AC produced is $x^2+y^2+2xy\cos A-2s(x+y)+s^2=0$ (i), AB, AC being axes. The radical centre of the escribed circle is given by $\frac{x}{a+c}+\frac{y}{b}=\frac{1}{2}$, $\frac{x}{c}+\frac{y}{a+b}=\frac{1}{2}$, its co-ordinates are therefore $\frac{c(a+c)}{4s}$ and $\frac{b(a+b)}{4s}$. The radius required is equal to the tangent to one of the circles from the radical centre, and this is found by substituting the co-ordinates in (i). 37. Let the centres of the circles be (x', y'), (x'', y''), the fixed points $(\pm a, 0)$, and the point of contact (x, y). Then we have (i) $(x'-a)^2+y'^2=c^2$, (ii) $(x''+a)^2+y''^2=c^2$, (iii) $(x'-x'')^2+(y'-y'')^2=4c^2$. Also $2x=x'+x''$, and $2y=y'+y''$. From (i) and (ii) $(x'-x'')(x'+x'')-2a(x'+x'')+(y'-y'')(y'+y'')=0$, i.e. $x(x'-x'')-2ax+y(y'-y'')=0$. This with (iii) gives us $(y'-y'')$ and $(x'-x'')$. Then, taking (iii) from twice the sum of (i) and (ii), we have $(x'+x'')^2+(y'+y'')^2+4a^2-4a(x'-x'')=0$; whence the required locus.

CHAPTER V.

4. The parabola is $y^2=\frac{(n+1)^2}{n}ax$, where $1 : n$ is the given ratio. 5. (i) a straight line through the vertex, (ii) the curve $y^2=nx^2+2ax$. 6 $y^2=x^2+6ax+a^2$. 10. The chord of contact of tangents from $(-4a, k)$ is $yk=2a(x-4a)$. The equation of the lines joining vertex to points

of contact is [Art. 38] $y^2 - 4ax\frac{2ax - yk}{8a^2} = 0$, or $y^2 - x^2 + \frac{k}{2a}xy = 0$. 11. $TN = \frac{y'^2 - 4ax'}{\sqrt{(y'^2 + 4a^2)}}$, and $TM^2 = y'^2 + 4a^2$; where (x', y') is T. 12. Let the axis of x be midway between the axes of the parabolas, then their equations will be $(y - b)^2 = 4ax$, $(y + b)^2 = 4ax$. If $y = \eta$ cut the curves in (x_1, η) and (x_2, η) respectively, we have $\eta^2 + b^2 = 2a(x_1 + x_2)$. Hence, if (ξ, η) be the middle point of intercept, $\eta^2 + b^2 = 4a\xi$. 15. The chord whose middle point is (x', y') is parallel to the polar of (x', y'); its equation therefore is $(y - y')y' = 2a(x - x')$. If the chord pass through the fixed point (h, k), we have $(k - y')y' = 2a(h - x')$. Hence the required locus is the parabola $y(y - k) = 2a(x - h)$. 25. Let $y = m_1x + \frac{a}{m_1}$...(i), $y = m_2x + \frac{a}{m_2}$...(ii), $y = m_3x + \frac{a}{m_3}$...(iii), and $y = m_4x + \frac{a}{m_4}$...(iv), be the equations of the four tangents. The ordinate of the point of intersection of (i) and (ii) is $a\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$, and the ordinate of the point of intersection of (iii) and (iv) is $a\left(\frac{1}{m_3} + \frac{1}{m_4}\right)$; hence the ordinate of the middle point of these intersections is $\frac{a}{2}\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4}\right)$. The symmetry of this result shews that the ordinate is the same for the middle point of the other *two* diagonals. 27. If the fixed line and the two tangents make angles α, θ_1, θ_2 with the axis, we have $2\alpha = \theta_1 + \theta_2$. And if (x', y') be the point of intersection of the tangents, $\tan\theta_1$ and $\tan\theta_2$ are the roots of $y' = mx' + \frac{a}{m}$. We therefore have $\tan 2\alpha = \frac{y'}{x' - a}$; which shews that the intersection of the tangents is on a fixed straight line; therefore, &c. 33. At points common to $y^2 - 4ax = 0$ and any circle $x^2 + y^2 + 2gx + 2fy + c = 0$, we have $\frac{y^4}{16a^2} + y^2 + 2g\frac{y^2}{4a} + 2fy + c = 0$. The coefficient of y^3 is zero, hence $y_1 + y_2 + y_3 + y_4 = 0$. If therefore the normals at y_1, y_2, y_3 meet in a point, y_4 is zero; for we know that $y_1 + y_2 + y_3 = 0$ [Art. 106]. 38. The normal at (x', y') is $2a(y - y') + y'(x - x') = 0$. If this pass through (h, k), we have $2ak + y'(h - x' - 2a) = 0$, whence $4a^2k^2 = 4ax'(h - x' - 2a)^2$. This gives a cubic equation for x' from which we have $x' + x'' + x''' = 2h - 4a$, or $(x' + a) + (x'' + a) + (x''' + a) + a = 2h$; therefore, &c. 41. Take for axes the tangent parallel to the given lines and the diameter through its point of contact. 43. The line is $x = 2a + c$. 44. The ordinates of the normals which meet in (h, k) are given by $2a(y - k) +$

$y(x-h)=0$. If (h, k) be on the curve, we have $2a(y-k)+\frac{y}{4a}(y^2-k^2)=0$. Hence the ordinates different from k are given by $y(y+k)+8a^2=0$; so that $y_1y_2=8a^2$. The equation of the chord is $y(y_1+y_2)-4ax-y_1y_2=0$, hence this cuts the axis where $x=-\frac{y_1y_2}{4a}=-2a$. 47. Let y_1, y_2, y_3, y_4 be the ordinates of the points A, B, C, D; and let AB, BC, CA, AD make angles θ_1, θ_2, θ_3, θ_4 with the axis. Then [Art. 102], $y_1+y_2=4a\cot\theta_1$, and so for the rest. Hence $\cot\theta_1+\cot\theta_3=\cot\theta_2+\cot\theta_4$; which shews that if three of the angles are constant, the fourth also is constant. 50. Let P, Q, R, S be (x_1, y_1) &c. The equation of the circle on PQ as diameter is $(y-y_1)(y-y_2)+(x-x_1)(x-x_2)=0$. Where this meets $y^2=4ax$, we have $(y-y_1)(y-y_2)+\frac{1}{16a^2}(y^2-y_1^2)(y^2-y_2^2)=0$. Hence y_3, y_4 are the roots of $16a^2+(y+y_1)(y+y_2)=0$, so that $y_3y_4=y_1y_2+16a^2$. But PQ, RS cut the axis at points whose abscissæ are $-\frac{y_1y_2}{4a}$ and $-\frac{y_3y_4}{4a}$, hence the difference of these abscissæ is $4a$.

CHAPTER VI.

4. The equation of a line through the middle point of a chord perpendicular to the chord can be written down by assuming Art. 114 (iii). 15. The ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{a^2-b^2}{a^2+b^2}\right)^2$. 18. Use $y=mx+\sqrt{(a^2m^2+b^2)}$. 20. Use eccentric angles. 21. The line $y=m(x-ae)$ cuts the ellipse where $\frac{x^2}{a^2}+\frac{m^2(x-ae)^2}{b^2}=1$. Put $x=\frac{1}{2}\left(ae+\frac{a}{e}\right)-p$, and shew that the product of the roots of the quadratic in p is independent of m. 27. If P be (x', y'), the point of intersection of QH and RS is $\left(-x', -y'\frac{1-e^2}{1+e^2}\right)$. 30. The semi-axes of the locus of P are the semi-sum and semi-difference of the radii of the circles. 36. The chord which has (x', y') for middle point is parallel to the polar of (x', y'), its equation is therefore $(x-x')\frac{x'}{a^2}+(y-y')\frac{y'}{b^2}=0$. Hence, if the chord pass through a fixed point (h, k), the middle point is on the ellipse $(h-x)\frac{x}{a^2}+(k-y)\frac{y}{b^2}=0$. 37. Let P be (x', y'), and let the chord make an angle θ with the major axis of the ellipse. The co-ordinates of the point on the chord at a distance r from

P are $x'+r\cos\theta$, and $y'+r\sin\theta$; substitute these co-ordinates in the equation of the ellipse for PQ, and in the equation $\frac{xx'}{a^2}+\frac{yy'}{b^2}=0$ for PR; then see Art. 111. 38. The equation of the locus is $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)=\frac{c^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$. 39. The equation is $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2+y^2}{a^2+b^2}$. 40. Let $\alpha, \beta, \gamma, \delta$ be the co-ordinates of the angular points A, B, C, D of the quadrilateral; then, since AB, BC, CD are parallel to three fixed straight lines, we have $(\alpha+\beta)$, $(\beta+\gamma)$ and $(\gamma+\delta)$ constant; therefore $(\alpha+\delta)$ is constant. 43. Let the co-ordinates of Q be $a\cos\theta$ and $b\sin\theta$, then quadrilateral $OPCQ=2\,.\,\text{triangle }OCQ=hb\sin\theta-ka\cos\theta=A$, suppose. Therefore $\frac{A}{ab}=\frac{h}{a}\sin\theta-\frac{k}{b}\cos\theta$ (1). But (h, k) is on the tangent at Q; therefore $\frac{h}{a}\cos\theta+\frac{k}{b}\sin\theta=1$ (2). From (1) and (2) we have $\frac{A^2}{a^2b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}-1$. The area of the triangle PCQ can be readily deduced from that of the quadrilateral. 50. Ans. $2(b^2y^2+a^2x^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2$. 53. If ϕ be the eccentric angle of P, the co-ordinates of Q are $(a+b)\cos\phi$ and $(a+b)\sin\phi$, or $(a-b)\cos\phi$, $(b-a)\sin\phi$, according as PQ is measured along the normal outwards or inwards. 55. Let T be (x', y'). In the quadratic equation giving the abscissæ of points where $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$ cuts the ellipse, substitute $\frac{r-a}{e}$ for x [Art. 110]; the product of the roots of the equation in r will be equal to $SP\,.\,SQ$.

CHAPTER VII.

3. An hyperbola. 4. An hyperbola. 5. A rectangular hyperbola. 19. $2y+3x+4=0$. 20. $x-2=0$, $y-3=0$, $xy-3x-2y+12=0$. 26. The lines joining $(x'y')$ to the two fixed points $(\pm a, 0)$ are $(yx'-xy')^2=a^2(y-y')^2$. These are parallel to $y^2(x'^2-a^2)-2x'y'xy+y'^2x^2=0$, the bisectors of which are $\frac{x^2-y^2}{y'^2-x'^2+a^2}+\frac{xy}{x'y'}=0$. Since these bisectors are fixed lines, we have $\frac{y'^2-x'^2+a^2}{x'y'}=\text{const.}$

CHAPTER VIII.

4. If α, β, γ be the vectorial angles of A, B, C respectively, $SA = \dfrac{l}{2\cos^2\dfrac{\alpha}{2}}$, and $SA' = \dfrac{l}{2\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}}$, &c. 5. As in Art. 165 (5), the perpendicular on the tangent at a makes with the axis an angle $\tan^{-1}\dfrac{\sin\alpha}{e+\cos\alpha}$. Therefore, &c. 7. See Art. 165 (3). 10. If the conics are $\dfrac{l}{r} = 1 + e\cos\theta$, and $\dfrac{l'}{r} = 1 + e'\cos(\theta-\alpha)$, the common chords are $\dfrac{l}{r} - e\cos\theta = \pm\left\{\dfrac{l'}{r} - e'\cos(\theta-\alpha)\right\}$. 13. If the conics are $\dfrac{l}{r} = 1 + e\cos\theta$, and $\dfrac{l'}{r} = 1 + e'\cos(\theta-\alpha)$, the common chords are $\dfrac{l \pm l'}{r} = e\cos\theta \pm e'\cos(\theta-\alpha)$. These touch respectively the conics $\dfrac{l \pm l'}{er} = 1 \pm \dfrac{e'}{e}\cos\theta$. 16. If d be the distance of the focus from the directrix, the conics will be $\dfrac{ed}{r} = 1 + e\cos\theta$, and $\dfrac{e'd}{r} = 1 + e'\cos(\theta-\alpha)$. If the conics touch one another at some point β, the equations $\dfrac{ed}{r} = e\cos\theta + \cos(\theta-\beta)$, and $\dfrac{e'd}{r} = e'\cos(\theta-\alpha) + \cos(\theta-\beta)$ will represent the same straight line. Write the equations in the forms $\dfrac{d}{r} = \cos\theta\left(1 + \dfrac{\cos\beta}{e}\right) + \sin\theta\,\dfrac{\sin\beta}{e}$, and $\dfrac{d}{r} = \cos\theta\left(\cos\alpha + \dfrac{\cos\beta}{e'}\right) + \sin\theta\left(\sin\alpha + \dfrac{\sin\beta}{e'}\right)$; equate the coefficients of $\cos\theta$, and of $\sin\theta$, and eliminate β. 17. Let the equation of the circle be $r = a\cos(\theta-\alpha)$, and the equation of the conic $\dfrac{l}{r} = 1 + e\cos\theta$. Eliminate θ, and we obtain a biquadratic for r.

CHAPTER IX.

7. Ans. $\lambda = 1$. 8. Ans. $10x^2 + 21xy + 9y^2 - 41x - 39y + 4 = 0$. 9. Ans. $3x^2 - 2xy - 5y^2 + 7x - 9y + 2 = 0$, and $3x^2 - 2xy - 5y^2 + 7x - 9y + 20 = 0$. 10. Ans. $6x^2 - 7xy - 3y^2 - 2x - 8y - 4 = 0$, and $6x^2 - 7xy - 3y^2 - 2x - 8y - 2 = 0$. 14. Take O for origin, and the axis of x through the centre of the circle. The equation of the circle will be $r = d\cos\theta$ (1); the equation of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, or in polars $ar^2\cos^2\theta$

$+ 2hr^2 \cos\theta \sin\theta + br^2 \sin^2\theta + 2gr\cos\theta + 2fr\sin\theta + c = 0$ (2). Eliminate θ from (1) and (2), then we obtain an equation in r the product of the four roots of which will be $\frac{cd^2}{(a-b)^2+4h^2}$. Since the origin is fixed, c is constant; and $(a-b)^2+4h^2$ is constant from Ex. 11.

CHAPTER X.

3. To find the fixed point in Ex. 1, take OP, OQ parallel to the axes; then PQ is a diameter, and CO, PQ make equal angles with the axis. Hence the co-ordinates of the point can be found referred to the centre and axes of the conic. The fixed point in Ex. 2 is the point where the tangent at O is met by the tangent at the other extremity of the normal through O, as is seen by taking OP, OQ indefinitely near to the normal. For locus see Art. 138 (4). 7. Take O for origin and the chords for axes. We have to prove that $\frac{a+b}{c}$ is independent of the *direction* of the axes. 13. In the parabola $y'y''$ is constant. 20. Take O for origin, the chord and its conjugate for axes; then the equation of the curve will be $ax^2+by^2+2fy+c=0$. Tangents from (x', y') are given by $\phi(x, y)\,\phi(x', y') - \{axx'+byy'+f(y+y')+c\}^2=0$; in this put $y=0$; then the coefficient of x will be zero if $fy'+c=0$, that is if (x', y') be on the polar of O. Or, let the tangents at P, Q meet in K; then KL, the polar of O, is parallel to AB; and if QOP meet the polar of O in L, $\{QOPL\}$ is harmonic. Hence $\{TOS\,\infty\}$ is harmonic, and therefore $TO=OS$. 22. (i) a conic; (ii) a straight line. 25. A curve of the fourth degree. 28. Corresponding to any point T on the tangent at P there is *one* point T' such that T, T' are equidistant from the centre, and there is *one* intersection of the tangents at T, T'; hence every tangent to the ellipse cuts the locus in one and only one point: the locus is therefore a *straight line*. If T, T' are on the director circle, the tangents from T, T' are parallel; therefore the direction of the point at infinity on the locus is perpendicular to the tangent at P; also when T, T' are both at infinity, the tangents from T, T' are parallel to the tangent at P, and therefore intersect at the extremity of the diameter through P, which proves the proposition. 37. The centre of the conic is given. Hence, if P be the given point, P', the other extremity of the diameter through P, is on all the conics. The locus is such that $SP \,.\, SP'$ is constant; this curve is called a lemniscate. 40. Let S, S' be the foci, S being given, C the centre, P the given point, and O the middle point of SP. Then $CD^2 = SP \,.\, S'P = 4SO \,.\, OC$. This proves that

the locus of D is a parabola, since CD and OC are drawn in fixed directions, and SO is fixed. 43. Let the variable ellipse touch at P, and let the tangents at S, P meet in T. CT bisects SP in V, and is therefore parallel to $S'P$, so that CT and SP make equal angles with the tangent at P; hence $VT=VP=VS$; therefore STP is a right angle, and CT is the radius of the director-circle of the variable ellipse. Hence, since $CT=\frac{1}{2}(SP+S'P)=$ constant, the question is reduced to 37. 44. This follows from Ex. 23, Chapter VII. 47. $\{PGOG'\}$ is harmonic, and GCG' is a right angle; therefore CP and CO make equal angles with CG. Then see solution of 2. 53. Ans. $c=\pm ab$. 54. Let the conic which goes through A, B, C, D, E cut the circle ABE in G; then, AB and CD make equal angles with the axes, and so also do AB and EG; hence EG is parallel to CD, so that G and F are coincident. The *direction* of the axes is known, we have therefore only to find the centre. If V, V' are the middle points of CD and EF respectively, VV' is a diameter. Draw a circle through D, C, E: if this cut the conic in a fourth point H, EH and CD make equal angles with the axes of the conic; therefore EH is parallel to AB; hence the line through the middle points of AB and EH is another diameter. Thus the centre is found. 55. The six points are always on a *conic*, and the conic is $(ax'x''+by'y''-1)\,(ax^2+by^2-1)-(axx'+byy'-1)\,(axx''+byy''-1)=0$ [see Ex. 3, Art. 187]. The conditions for a circle are $x'x''-y'y''=\frac{1}{a}-\frac{1}{b}$ (1), and $x'y''+x''y'=0$ (2). Square and add, then $(x'^2+y'^2)\,(x''^2+y''^2)=\left(\frac{1}{b}-\frac{1}{a}\right)^2$, that is $CP\,.\,CP'=CS^2$, where C is the centre and S is a focus; also from (2) CP and CP' make equal angles with the axis of x; and (1) and (2) shew that P, P' are on different sides of the transverse axis. When the curve is a parabola P, P' are on a line through the focus, and equidistant from the focus. 58. The chord of $ax^2+by^2-1=0$ which has (x', y') for middle point is parallel to the polar of (x', y') and its equation is $(x-x')\,ax'+(y-y')\,by'=0$. The line through (x', y') perpendicular to the chord must pass through $O\,(f, g)$; hence we have $\frac{f-x'}{ax'}=\frac{g-y'}{by'}$, so that (x', y') is on a rectangular hyperbola. 59. Any conic of the system is given by $ax^2+by^2-1-\lambda\,\{(x-\alpha)^2+(y-\beta)^2-c^2\}=0$, where (α, β) is the point O. Find the centre, and eliminate λ. 63. If the normal to $ax^2+by^2-1=0$ at $P\,(x',y')$ pass through $O\,(f, g)$ we have $\frac{f-x'}{ax'}=\frac{g-y'}{by'}$, or $fby'-agx'+(a-b)\,x'y'=0$ (1). We have to shew that $\frac{x-x'}{x'}+\frac{y-y'}{y'}=0$, or $xy'+yx'-2x'y'=0$, will go through the same point if (x',y') is any one of the four points of intersection of (1) and the conic. The point is $\frac{x}{fb}=\frac{y}{-ag}=\frac{2}{b-a}$. 73. A conic. 75. The four points are $\left(\frac{a^2}{x'},\frac{b^2}{y'}\right)$ &c., where $(x'y')$ &c. are the feet of the normals. Now, if (f, g) be the point at which the normals meet, $\frac{a^2f}{x'}-\frac{b^2g}{y'}=a^2-b^2$. Hence $\left(\frac{a^2}{x'},\frac{b^2}{y'}\right)$ is on the straight line $fx-gy=a^2-b^2$, and so also are the other three points. 83. If $y=m(x-ae)$ be the chord, the circle is $x^2-a^2+y^2+2maey-m^2b^2=0$, or $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1-\frac{1}{a^2b^2}(aey-mb^2)^2=0$. 96. If $xy=c^2$ be the equation of the hyperbola, and (x_1, y_1) &c. be the four points, and (α, β) be P; then $PA\,.\,Pa=\frac{\alpha^2(y_1-\beta)(y_2-\beta)(y_3-\beta)(y_4-\beta)}{c^4+y_1y_2y_3y_4}$.

CHAPTER XI.

3. Let the equation of the conic which passes through O be $ax^2+2hxy+by^2+2fy=0$, the tangent and normal at O being axes. If $a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$ be the equation of another conic, all the conics through their common points are included in $ax^2+2hxy+by^2+2fy+\lambda(a'x^2+2h'xy+b'y^2+2g'x+2f'y+c')=0$. Put $y=0$, then $\frac{1}{x_1}+\frac{1}{x_2}=-\frac{2g'}{c'}$, and therefore is independent of λ. 5. The axes of the parabolas are always parallel to conjugate diameters [Art. 207]. Now in a given ellipse the acute angle between two conjugate diameters is least when they are the equi-conjugates; and in different ellipses the angle between the equi-conjugates is greatest in that which has the least eccentricity. Hence if a pair of conjugate diameters are known, the conic has the least eccentricity when they are the equi-conjugates. 6. If TQ, TQ' be the tangents, and V be the middle point of QQ', TV and QQ' are parallel to conjugate diameters. See solution to 5. 16. Use the result of Ex. 2, Art. 219. 21. A circle. 24. $\tan^2\frac{\theta}{2}=-\frac{\lambda_2}{\lambda_1}$, where θ is the angle between the tangents. 25. The result follows from Art. 229 and Art. 186, Cor. 1. 28. Art. 227. 35. If TO' be the other bisector of the angle QTP, then $T\{QOPO'\}$ is harmonic, and therefore TO' is the polar of O. Let ROR' cut TO' in K, then $T\{ROR'K\}$ is harmonic, and OTR is a right

angle; hence RT, $R'T$ make equal angles with OT. 37. and 38. Use 'If a circle cut a parabola in four points the *sum* of the distances of those points from the axis of the parabola is zero.' 42. Shew that the conic, with respect to which the triangle formed by $x=0$, $y=0$, and $lx+my+1=0$ is self-polar, is $ax^2+2lmxy+by^2+2lx+2my+1=0$. 50. Let the hyperbola be $2xy=c$, and the first circle $x^2+y^2+2gx+2fy=0$. Let (x_1, y_1) &c. be the four points. The equation of the second circle is $x^2+y^2+2xx_4+2yy_4=0$. The point of intersection of the tangents at B, C is $\left(2\frac{x_2x_3}{x_2+x_3}, \frac{c}{x_2+x_3}\right)$; this is on the second circle if $c^2(x_2+x_3+x_4)+4x_2x_3x_4(x_2x_3+x_3x_4+x_4x_2)=0$ (1). Now the equation giving the abscissæ of A, B, C, D is $4x^4+8gx^3+4fcx+c^2=0$. Hence $4x_1x_2x_3x_4=c^2$, and $x_1(x_2+x_3+x_4)+x_2x_3+x_3x_4+x_4x_2=0$; and these shew that (1) is true.

CHAPTER XII.

1. A parabola. 3. An hyperbola. 4. (1) A similar ellipse. (2) An ellipse. 12. A common chord, which is not a diameter, subtends a right angle at the centre. The envelope is a circle. 16. If the conic is $ax^2+by^2=1$, and c the radius of the circle, the envelope is $ax^2+by^2=\frac{abc^2}{a+b}$. The envelope is the original conic if $c^2=\frac{1}{a}+\frac{1}{b}$; that is, if the circle is the director-circle of the conic. 20. See Art. 197. 25. The equation of the envelope is $xy=\pm 4ab$. 26. If the original conic is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the envelope is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+2\frac{x}{a}=0$. 27. Take the fixed point for origin, and let the lines be $(x-a)(x-a')=0$; then the envelope is $(a-a')^2y^2=4aa'(x-a)(x-a')$. 31. Take the given diameters for axes, and let the conic be $ax^2+2hxy+by^2-1=0$; then the envelope is $4(ax+hy)(hx+by)=\frac{h}{h^2-ab}$. 38. Let the equation of the conic be $ax^2+by^2=1$, and let O be (α, β). Transfer the origin to O, and let $lx+my+1=0$ be the equation of PQ, one of the chords. Write down the equation of OP, OQ [Art. 38]; then the condition of perpendicularity gives the tangential equation, viz. $(l^2+m^2)(a\alpha^2+b\beta^2-1)-2a\alpha l-2b\beta m+a+b=0$. One focus is (0, 0), the centre is $\left(-\frac{a\alpha}{a+b}, -\frac{b\beta}{a+b}\right)$, and the other focus is $\left(-\frac{2a\alpha}{a+b}, -\frac{2b\beta}{a+b}\right)$. If $a : b$ is constant, the envelopes are confocal. If the given conic is a rectangular hyperbola $a+b=0$, and the envelope is a *parabola*.

N.B. The following are important special forms of the tangential equation $\phi(l, m)=0$.

(i) If $c=0$, the conic is a parabola.
(ii) If $a=b=0$, the conic touches the axes.
(iii) If $a=b$, and $h=2a\cos\omega$ the origin is a focus.

CHAPTER XIII.

2. Ans. $$\frac{abc}{8\Delta^2}\begin{vmatrix} \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \\ \alpha''', & \beta''', & \gamma''' \end{vmatrix}.$$

3. The four points of intersection of any two of the conics are of the form $\pm f$, $\pm g$, $\pm h$. The conic $u\alpha^2+v\beta^2+w\gamma^2=0$ will pass through the points $(\pm f, \pm g, \pm h)$, and $(\pm f', \pm g', \pm h')$ if $uf^2+vg^2+wh^2=0$, and $uf'^2+vg'^2+wh'^2=0$. 6. Let the lines be $l\alpha\pm m\beta\pm n\gamma=0$; then the two points on the diagonal $\alpha=0$ are given by $m^2\beta^2+2\lambda\beta\gamma+n^2\gamma^2=0$, $\alpha=0$. The other pairs are $n^2\gamma^2+2\mu\gamma\alpha+l^2\alpha^2=0$, $\beta=0$; and $l^2\alpha^2+2\nu\alpha\beta+m^2\beta^2=0$, $\gamma=0$. These are all on the conic $l^2\alpha^2+m^2\beta^2+n^2\gamma^2+2\lambda\beta\gamma+2\mu\gamma\alpha+2\nu\alpha\beta=0$.

7. The perpendicular distances of (α, β, γ) from the three sides are $\frac{1}{a}(a\alpha-b\beta-c\gamma)$, &c. Hence the equation required is

$$\frac{a^2}{b\beta+c\gamma-a\alpha}+\frac{b^2}{c\gamma+a\alpha-b\beta}+\frac{c^2}{a\alpha+b\beta-c\gamma}=0.$$

8. The equation of the circle is of the form $a\beta\gamma+b\gamma\alpha+c\alpha\beta+\lambda(a\alpha+b\beta+c\gamma)^2=0$. If this cut BC in P, P', then $BP \,.\, BP'=r^2-R^2$. 9. The point which is at a distance ρ from $(\alpha_0, \beta_0, \gamma_0)$, on a line parallel to BC is $(\alpha_0, \beta_0+\rho\sin C, \gamma_0-\rho\sin B)$. If this point be on the conic, we have

$$\rho_1\rho_2=\frac{l\beta_0\gamma_0+m\gamma_0\alpha_0+n\alpha_0\beta_0}{-l\sin B\sin C}.$$

Hence we have

$$\frac{r_1^2}{\frac{a}{l}}=\frac{r_2^2}{\frac{b}{m}}=\frac{r_3^2}{\frac{c}{n}}.$$

[If the conic were given by the general equation we should have

$$\rho_1\rho_2=\frac{\phi(\alpha_0, \beta_0, \gamma_0)}{v\sin^2 C+w\sin^2 B-2u'\sin B\sin C}.$$

Hence we find at once the conditions for a circle, viz. $vc^2+wb^2-2u'bc$ = similar expressions.]

11. If P be (f, g, h), K, L, M are on the line $\frac{\alpha}{f}+\frac{\beta}{g}+\frac{\gamma}{h}=0$. If P be on $l\alpha+m\beta+n\gamma=0$ KLM touches $\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0$ &c. 12. If O be (f, g, h) and O' be (f', g', h') then Z is given by $\frac{\alpha}{ff'(gh'-g'h)}$ $=\frac{\beta}{gg'(hf'-h'f)}=\frac{\gamma}{hh'(fg'-f'g)}$. If O, O' are on the fixed conic $\lambda\beta\gamma+\mu\gamma\alpha+\nu\alpha\beta=0$, Z is the fixed point (λ, μ, ν).

(Examples 11 and 12 are taken from an interesting paper by Rev. H. Martin published in the *Messenger of Mathematics*, Vol. IV.)

18. If $u\alpha^2+v\beta^2+w\gamma^2=0$, be a parabola it will touch the line at infinity, and therefore all the four lines given by $a\alpha\pm b\beta\pm c\gamma=0$. 20. If $(\alpha', \beta', \gamma')$ be one focus, the other will be $\left(\frac{1}{\alpha'}, \frac{1}{\beta'}, \frac{1}{\gamma'}\right)$; write down the condition that the fixed point (f, g, h) may be on the line joining these two points. 34. Let (f, g, h) be the point of intersection of AA', BB', CC', then A' is (f', g, h), B' is (f, g', h), and C' is (f, g, h'). BC', CB' intersect in A'' where $\frac{\alpha}{f}=\frac{\beta}{g'}=\frac{\gamma}{h'}$. Hence the equation of $A'\,A''$ is

$$\begin{vmatrix} \alpha & \beta & \gamma \\ f' & g & h \\ f & g' & h' \end{vmatrix}=0.$$

It is clear that $A'\,A''$, and also the other two diagonals $B'\,B''$, $C'\,C''$, of the hexagon formed by the six lines, pass through the point $(f+f', g+g', h+h')$. Hence by Brianchon's Theorem the hexagon circumscribes a conic. 40. Consider any two of the conics, and draw their fourth common tangent. Then, the radical axis of their director-circles is the directrix of the parabola touching the four lines [Art. 299 (5)]; the radical axis therefore [Art. 308, Ex. 4] passes through the orthocentre of the original triangle. Then, since the director-circles are equal, it follows that the centres of *any two* of the conics, and therefore the centres of all the conics, are equidistant from the orthocentre of the triangle. 41. The centre of the circle with respect to which the triangle is self-polar is the orthocentre. Hence, from 40, the theorem will be true for *all* conics whose director-circles are equal, if it be true for any one of them. Let ABC be the triangle, and O the orthocentre, and let OA cut BC in A'. Then, if P be any point on BC, the line AP is a limiting form of an inscribed conic, and the circle on AP as diameter is its director-circle; also $OA\,.\,OA'$ is equal to the square of the tangent to this circle from O, and $OA\,.\,OA'$ is equal to the square of the radius of the self-polar circle, hence the self-polar circle cuts the director circle at right angles. This proves the proposition, since P is any point on BC. 50. The equations of the tangents from the angular points of the fundamental triangle are $Vz^2+Wy^2-2U'yz=0$, &c. Hence the six points are on the conic

$$VWx^2+WUy^2+UVz^2-2UU'yz-2VV'zx-2WW'xy=0,$$

This intersects

$$U'^2x^2+V'^2y^2+W'z^2-2V'W'yz-2W'U'zx-2U'V'xy=0$$

in the same four points as

$$(VW-U'^2)\,x^2+\ldots\ldots\ldots+2\,(V'W'-UU')\,yz+\ldots\ldots\ldots=0.$$

But this latter conic is the original conic, since $VW-U'^2=u\Delta$, &c.

THE END.

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